Solutions to Exercises 5.2.2 through 5.2.11.

5.2.2. To show that $\mathcal{U}(\theta, \theta+1)$ has monotone likelihood ratio, take $\theta_1 < \theta_2$ and consider two cases. Case 1: $\theta_1 + 1 < \theta_2$. The likelihood ratio is

$$L(\theta_1, \theta_2) = \frac{f(x|\theta_1)}{f(x|\theta_2)} = \begin{cases} 0, & \text{if } \theta_1 < x < \theta_1 + 1; \\ \infty, & \text{if } \theta_2 < x < \theta_2 + 1; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Case 2: $\theta_1 < \theta_2 < \theta_1 + 1$.

$$L(\theta_1, \theta_2) = \frac{f(x|\theta_1)}{f(x|\theta_2)} = \begin{cases} 0, & \text{if } \theta_1 < x < \theta_2; \\ 1, & \text{if } \theta_2 < x < \theta_1 + 1; \\ \infty, & \text{if } \theta_1 + 1 < x < \theta_2 + 1; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

In either case, the likelihood ratio is nondecreasing on the set where it is defined. (This analysis is for a sample of size one. The concept of monotone likelihood ratio is a one dimensional concept. The MLR theory cannot be applied for a sample of size n, because sufficiency only reduces the problem to a two-dimensional sufficient statistic, $(\min X_i, \max X_i)$.)

Here is a counterexample to the second statement of Theorem 5.2.1 when the size of the test is zero. In the $\mathcal{U}(\theta, \theta+1)$ problem above, the test $\phi_1(x) = \mathrm{I}(x > \theta_0 + 2)$ is a test of the form (5.24) and it has size zero. But the test $\phi_2(x) = \mathrm{I}(x > \theta_0 + 1)$ also has size zero and is better than ϕ_1. It has strictly greater power for $\theta_0 < \theta < \theta_0 + 2$.

5.2.3. Fix $\theta_1 < \theta_2$. Then

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \begin{cases} c(\theta_2)/c(\theta_1) & \text{if } x < \theta_1 \\ +\infty & \text{if } \theta_1 < x < \theta_2 \\ \text{undefined} & \text{if } x > \theta_2. \end{cases}$$

Since this is non-decreasing in x for $x < \theta_2$, this family has monotone likelihood ratio. Note that the class of uniform distributions, $\mathcal{U}(0, \theta)$, is a special case of this family, with $c(\theta) = 1/\theta$ and $h(x) = \mathrm{I}_{(0,\infty)}(x)$.

5.2.4. Let $f(x|\theta) = \exp\{-|x-\theta|/\beta\}/(2\beta)$ be the density of X, where β is known. For fixed $\theta_1 < \theta_2$,

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \begin{cases} e^{-(\theta_2-\theta_1)/\beta} & \text{if } x < \theta_1 \\ e^{(2x-\theta_2-\theta_1)/\beta} & \text{if } \theta_1 < x < \theta_2 \\ e^{(\theta_2-\theta_1)/\beta} & \text{if } x > \theta_2. \end{cases}$$

This is nondecreasing in x, so the family has monotone likelihood ratio.

5.2.5. The density is $f(x|\theta) = e^{-(x-\theta)}\mathrm{I}_{(\theta,\infty)}(x)$. Let $\theta_1 < \theta_2$. Then

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{e^{-(x-\theta_2)}\mathrm{I}_{(\theta_2,\infty)}(x)}{e^{-(x-\theta_1)}\mathrm{I}_{(\theta_1,\infty)}(x)} = \begin{cases} \text{undefined} & \text{if } x \le \theta_1 \\ 0 & \text{if } \theta_1 < x \le \theta_2 \\ e^{\theta_2-\theta_1} & \text{if } \theta_2 < x. \end{cases}$$

This is clearly nondecreasing on its domain of definition, which is $x > \theta_1$.

5.2.6. The Cauchy distribution, $\mathcal{C}(0, \theta)$, has density $f(x|\theta) = \dfrac{\theta}{\pi(\theta^2 + x^2)}$. If $0 < \theta_1 < \theta_2$, then

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{\theta_2(\theta_1^2 + x^2)}{\theta_1(\theta_2^2 + x^2)}.$$

This is defined for all x, but it has a minimum at $x = 0$, is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. So it does not have monotone likelihood ratio.

But if $T = |X|$, then the density of T is $f_T(t|\theta) = \dfrac{2\theta}{\pi(\theta^2 + t^2)} \mathrm{I}_{[0,\infty)}(t)$. For $0 < \theta_1 < \theta_2$, we have

$$\frac{f_T(t|\theta_2)}{f_T(t|\theta_1)} = \frac{\theta_2(\theta_1^2 + t^2)}{\theta_1(\theta_2^2 + t^2)},$$

on its domain of definition, which is $[0, \infty)$. On this interval, the ratio is increasing.

5.2.7. (a) The distribution of $T = \max(X_i)$ has density

$$f_T(t|\theta) = nt^{n-1}\theta^{-n}\mathrm{I}(0 < t < \theta)$$

The class of Neyman-Pearson best tests have the form given in (5.7) for some $k \geq 0$ or (5.8). In our case, this reduces to the class of tests

$$\phi(t) = \begin{cases} 1 & \text{if } \theta_0 < t < \theta_1 \\ \gamma(t) & \text{if } t \leq \theta_0 \text{ or } t \geq \theta_1 \end{cases} \tag{1}$$

where $0 \leq \gamma(t) \leq 1$ is arbitrary and determines the size of the test. Every best test is of this form (up to a set of probability zero), and each of these tests is best of its size.

(b) A test, $\phi(t)$, that is in this class for all $\theta_1 > \theta_0$ is UMP of its size for testing $H_0 : \theta = \theta_0$ against $H_1' : \theta_1 > \theta_0$. This is the class of tests,

$$\phi(t) = \begin{cases} 1 & \text{if } t > \theta_0 \\ \gamma(t) & \text{if } t \leq \theta_0 \end{cases} \tag{2}$$

for arbitrary $0 \leq \gamma(t) \leq 1$.

(c) The test

$$\phi(t) = \begin{cases} 1 & \text{if } t > \theta_0 \\ \alpha & \text{if } t \leq \theta_0 \end{cases}$$

is in this class and so is UMP for testing H_0 against H_1'. Moreover, the power function, $\mathrm{E}_\theta\phi(T)$ is a constant, α, for $\theta \leq \theta_0$, and so this test is of size α for testing $H_0' : \theta \leq \theta_0$. Since such tests form a subclass of the tests of size α for testing H_0, this test is also UMP for testing H_0' against H_1'.

Since this test is UMP, we cannot improve on the power at any $\theta > \theta_0$ without decreasing the size at $\theta = \theta_0$. However, the test,

$$\phi_1(t) = \begin{cases} 1 & \text{if } t > (1 - \sqrt[n]{\alpha})\theta_0 \\ 0 & \text{otherwise} \end{cases}$$

is also UMP of size α for testing H_0' vs H_1', but for $0 < \alpha < 1$ the size is smaller than α for all $\theta < \theta_0$. (In fact the size is zero for $\theta < (1 - \sqrt[n]{\alpha})\theta_0$.) Thus, if $0 < \alpha < 1$, ϕ is not admissible since ϕ_1 is better.

(d) The class of best tests for testing $\theta = \theta_0$ vs $\theta = \theta_1$ for $\theta_1 < \theta_0$ are tests of one of the forms

$$\phi(t) = \begin{cases} 0 & \text{for } \theta_1 < t < \theta_0 \\ \gamma(t) & \text{otherwise.} \end{cases} \quad \text{or} \quad \phi(t) = \begin{cases} 1 & \text{if } t < \theta_1 \\ \gamma(t) & \text{otherwise} \end{cases} \tag{3}$$

For every $\theta_1 < \theta_0$, the test

$$\phi_2(t) = \begin{cases} 1 & \text{if } t > \theta_0 \text{ or } t < b = \theta_0 \sqrt[n]{\alpha} \\ 0 & \text{if } b < t < \theta_0 \end{cases}$$

is of one of these two forms, and, in addition, is of the form found in part (b). Thus, ϕ_2 is UMP of its size for testing H_0 against the two-sided hypothesis, $H_2' : \theta < \theta_0$. It is easy to check that this test has size α.

(e) For use later in Exercise 5.8.7, we generalize (5.30) slightly to

$$f(x|\theta) = c(\theta)h(x)\mathrm{I}_{(-\infty,\pi(\theta))}(x)$$

where $\pi(\theta)$ is an increasing function of θ. Then the distribution of $T = \max\{X_1, \ldots, X_n\}$ has density

$$f_T(t|\theta) = c(\theta)^n n \left(\int_{-\infty}^{t} h(x)\,dx\right)^{n-1} h(t)\mathrm{I}(t < \pi(\theta))$$

The class of Neyman-Pearson best tests of $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$, consists of tests of form (1) with θ_0 and θ_1 replaced by $\pi(\theta_0)$ and $\pi(\theta_1)$ respectively. The UMP tests of H_0 against $H_1' : \theta > \theta_0$ are still those tests of the form (2) with θ_0 replaced by $\pi(\theta_0)$. The best tests of $\theta = \theta_0$ against $\theta = \theta_1$ for $\theta_1 < \theta_0$ are of the form (3) with θ_0 and θ_1 replaced by $\pi(\theta_0)$ and $\pi(\theta_1)$ respectively. The test

$$\phi_2(t) = \begin{cases} 1 & \text{if } t > \pi(\theta_0) \text{ or } t < b \\ 0 & \text{if } b < t < \pi(\theta_0) \end{cases}$$

where $b < \pi(\theta_0)$, is of both forms and so is UMP of its size for testing H_0 against $H : \theta \neq \theta_0$. To achieve size α, b must satisfy $c(\theta_0)\int_{-\infty}^{b} h(x)\,dx = \sqrt[n]{\alpha}$.

5.2.8. (a)

$$\begin{aligned} R(\text{bluff}, \phi) &= P[(a+b)\mathrm{E}_{\lambda_1}(1-\phi(X)) + a\mathrm{E}_{\lambda_1}\phi(X)] + (1-P)[-(a+b)\mathrm{E}_{\lambda_0}(1-\phi(X)) + a\mathrm{E}_{\lambda_0}\phi(X)] \\ &= (2P-1)(a+b) - Pb\mathrm{E}_{\lambda_1}\phi(X) + (1-P)(2a+b)\mathrm{E}_{\lambda_0}\phi(X) \\ R(\text{honest}, \phi) &= P[(a+b)\mathrm{E}_{\lambda_1}(1-\phi(X)) + a\mathrm{E}_{\lambda_1}\phi(X)] - a(1-P) \\ &= a(2P-1) + bP - bP\mathrm{E}_{\lambda_1}\phi(X) \end{aligned}$$

(b) The minimax rule is that ϕ that minimizes the maximum of $R(\text{bluff}, \phi)$ and $R(\text{honest}, \phi)$. For a fixed $\mathrm{E}_{\lambda_0}\phi(X)$, both $R(\text{bluff}, \phi)$ and $R(\text{honest}, \phi)$ are minimized by maximizing $\mathrm{E}_{\lambda_1}\phi(X)$, so that the minimax rule must be a best test of its size for testing H_0 against H_1. Thus, if ϕ_α denotes a best test of size α, we restrict attention to the class of tests ϕ_α and find that α that minimizes the maximum risk. It is easy to check that if $\mathrm{E}_{\lambda_0}\phi_\alpha(X) = \alpha = b/(2a+b)$, then $R(\text{bluff}, \phi_\alpha) = R(\text{honest}, \phi_\alpha) = V$, say. Moreover, since $\mathrm{E}_{\lambda_1}\phi_\alpha(X)$ is increasing in α, $R(\text{honest}, \phi_\alpha)$ is decreasing in α so that $R(\text{honest}, \phi_\alpha) \geq V$ for $\alpha \leq b/(2a+b)$. This rule will therefore be minimax if we show that $R(\text{bluff}, \phi_\alpha) \geq V$ for $\alpha \geq b/(2a+b)$.

The general argument of Lemma 1.7.1 shows that $g_1(\alpha) = R(\text{bluff}, \phi_\alpha)$ is convex in α: If ϕ_0 achieves the minimum of $R(\text{bluff}, \phi_{\alpha_0})$ and ϕ_1 achieves the minimum of $R(\text{bluff}, \phi_{\alpha_1})$, then for arbitrary $\pi \in (0,1)$, the rule ϕ that uses ϕ_0 with probability π and ϕ_1 with probability $1-\pi$ has size $\alpha = \pi\alpha_0 + (1-\pi)\alpha_1$ and $R(\text{bluff}, \phi) = \pi R(\text{bluff}, \phi_{\alpha_0}) + (1-\pi)R(\text{bluff}, \phi_{\alpha_1})$. But the optimal rule can do at least this well; that is $R(\text{bluff}, \phi_{\pi\alpha_0+(1-\pi)\alpha_1}) \leq \pi R(\text{bluff}, \phi_{\alpha_0}) + (1-\pi)R(\text{bluff}, \phi_{\alpha_1})$.

There remains to show that $g_1(0) \leq g_1(b/(2a+b))$. Since $g_1(0) = (2P-1)(a+b) - Pb\mathrm{E}_{\lambda_1}\phi_0(X)$ and $g_1(b/(2a+b)) = (2P-1)(a+b) - Pb\mathrm{E}_{\lambda_1}\phi_{b/(2a+b)}(X) + (1-P)(2a+b)b/(2a+b)$, we have $g_1(0) \leq g_1(b/(2a+b))$ if and only if $\mathrm{E}_{\lambda_1}\phi_{b/(2a+b)}(X) - \mathrm{E}_{\lambda_1}\phi_0(X) \leq (1-P)/P$. If $P < 1/2$, this is always true.

It is interesting to note that II's optimal strategy is independent of P and λ_1. It is exactly the UMP test of H_0 vs $H_1' : \lambda < \lambda_0$. On the other hand, Player I's optimal strategy does depend on these quantities. This strategy is to bluff with probability $1 - g_1'(b/(2a+b))/((1-P)(2a+b))$, where g_1' represents a derivative of g_1 in the sense of being the slope of any supporting hyperplane of the graph of g_1.

5.2.9. Let $\theta_1 < \theta_2$. Then,

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{(1+e^{x-\theta_1})^2}{(1+e^{x-\theta_2})^2} e^{\theta_1-\theta_2} = \left(\frac{1+cy}{1+y}\right)^2 \frac{1}{c}$$

where $y = e^{x-\theta_1}$ and $c = e^{\theta_2-\theta_1}$. It is easy to see that this ratio is increasing in y since $c > 1$. But since y is increasing in x, the ratio is also increasing in x.

5.2.10. Given $f(x|\theta) = c(\theta)h(x)\mathrm{I}_{(\pi_1(\theta),\pi_2(\theta))}(x)$, with $\pi_1(\theta) < \pi_2(\theta)$, both nondecreasing in θ, we compute the likelihood ratio for fixed $\theta_1 < \theta_2$ in two cases. First, if $\pi_1(\theta_2) < \pi_2(\theta_1)$ then

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \begin{cases} 0 & \text{if } \pi_1(\theta_1) < x < \pi_1(\theta_2) \\ c(\theta_2)/c(\theta_1) & \text{if } \pi_1(\theta_2) < x < \pi_2(\theta_1) \\ +\infty & \text{if } \pi_2(\theta_1) < x < \pi_2(\theta_2) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Second, if $\pi_1(\theta_2) \geq \pi_2(\theta_1)$ then

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \begin{cases} 0 & \text{if } \pi_1(\theta_1) < x < \pi_2(\theta_1) \\ +\infty & \text{if } \pi_1(\theta_2) < x < \pi_1(\theta_2) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

In either case, the ratio is nondecreasing in x in the domain of definition of the ratio.

5.2.11. The likelihood ratio for testing $H_0' : \theta = 0$ against a simple alternative $\theta > 0$ is

$$f(x_1, \ldots, x_n|\theta)/f(x_1, \ldots, x_n|0) = \frac{\mathrm{I}(\theta < t_1 < t_2 < \theta + 1)}{\mathrm{I}(0 < t_1 < t_2 < 1)} = \begin{cases} 0 & \text{if } t_1 < \theta \\ 1 & \text{if } \theta < t_1 < t_2 < 1 \\ \infty & \text{if } t_2 > 1 \end{cases}.$$

The class of best tests of H_0' against a fixed $\theta > 0$ consists of the tests of the two forms

$$\phi(t_1, t_2) = \begin{cases} 1 & \text{if } t_2 > 1 \\ \text{any} & \text{if } \theta < t_1 < t_2 < 1 \\ 0 & \text{if } t_1 < \theta \end{cases} \qquad \text{or} \qquad \phi(t_1, t_2) = \begin{cases} 1 & \text{if } \theta < t_1 \\ \text{any} & \text{if } t_1 < \theta. \end{cases}$$

The tests of this form for all $\theta > 0$ are

$$\phi_0(t_1, t_2) = \begin{cases} 1 & \text{if } t_1 > k \text{ or } t_2 > 1 \\ 0 & \text{if } t_1 < k \text{ and } t_2 < 1 \end{cases}$$

for some $k \geq 0$. These are the UMP tests for testing H_0' against $H_1 : \theta > 0$. To show they are UMP for testing H_0 against H_1, we use the argument of the text: these tests have nondecreasing power function on the set $(-\infty, 0]$ and so are UMP out of the smaller class of tests that have size no greater than α on $(-\infty, 0]$. To find k to achieve a given α, note $\alpha = \mathrm{P}_0(T_1 > k) = (1-k)^n$, so that $k = 1 - \alpha^{1/n}$.