Solutions to the Exercises of Section 5.1.

5.1.1. Let us use the notation, $\alpha_0(\phi) = R(\theta_0, \phi) = \mathrm{E}_{\theta_0}(\phi(X))$ and $\alpha_1(\phi) = R(\theta_1, \phi) = 1 - \mathrm{E}_{\theta_1}(\phi(X))$. We are given that ϕ is not admissible. This means that there is a test ϕ' better than ϕ, which means that both $\alpha_0(\phi') \le \alpha_0(\phi)$ and $\alpha_1(\phi') \le \alpha_1(\phi)$ with at least one a strict inequality. But since ϕ is best of size α_0 and $\alpha_0(\phi') \le \alpha_0(\phi)$, we must have $\alpha_1(\phi') \ge \alpha_1(\phi)$. Hence, $\alpha_1(\phi') = \alpha_1(\phi)$, and therefore, $\alpha_0(\phi') < \alpha_0(\phi)$. We are to show that $\alpha_1(\phi)$ cannot be positive.

If $\alpha_1(\phi) > 0$, define a new test $\phi'' = \lambda\phi' + 1 - \lambda$, where λ is chosen so that $\alpha_0(\phi'') = \alpha_0(\phi)$ i.e. $\lambda\alpha_0(\phi') + 1 - \lambda = \alpha_0(\phi)$. This gives $\lambda = (1 - \alpha_0(\phi))/(\alpha_0(\phi) - \alpha_0(\phi'))$, so that $0 \le \lambda < 1$. But then

$$\alpha_1(\phi'') = 1 - \mathrm{E}_{\theta_1}(\phi'(X)) - 1 + \lambda = \lambda\alpha_1(\phi') = \lambda\alpha_1(\phi) < \alpha_1(\phi).$$

Thus, if $\alpha_1(\phi) > 0$ we have $\alpha_1(\phi'') < \alpha_1(\phi)$ which contradicts the assumption that ϕ is a best test of size α_0.

5.1.2. Let ϕ_0 be of the given form and let $0 \le \phi \le 1$ be any other function. Then,

$$\int (\phi_0(x) - \phi(x))(f_0(x) - \sum_1^n k_j f_j(x))\, dx \ge 0,$$

since the integrand is nonnegative from the definition of ϕ_0. Hence,

$$0 \le \int \phi_0(x) f_0(x)\, dx - \int \phi(x) f_0(x)\, dx - \sum_1^n k_j \int (\phi_0(x) - \phi(x)) f_j(x)\, dx. \tag{1}$$

If

$$\int \phi(x) f_j(x)\, dx = \int \phi_0(x) f_j(x)\, dx \quad \text{for} \quad j = 1, \ldots, n,$$

then each term of the summation in (1) is zero so that

$$\int \phi(x) f_0(x)\, dx \le \int \phi_0(x) f_0(x)\, dx \tag{2}$$

as was to be shown. If $k_j \ge 0$ for all j, and if

$$\int \phi(x) f_j(x)\, dx \le \int \phi_0(x) f_j(x)\, dx \quad \text{for} \quad j = 1, \ldots, n,$$

then each term of the summation in (1) is nonnegative, so we still have (2).

5.1.3. The likelihood ratio, $f_1(x)/f_0(x)$ takes on three values 0, 1 and ∞, with probabilities $1/2$, $1/2$ and 0 under H_0 and probabilities 0, $1/2$, and $1/2$ under H_1. Rejecting H_0 if $X > 1$ gives the point $(0, 1/2)$ in the risk set. Rejecting H_0 if $1/2 < X < 1$ gives the point $(1/2, 0)$ in the risk set. This gives the lower boundary of the risk set S to be the lines from $(0, 1/2)$ to $(1/2, 0)$. The risk set is given in Figure 1.

5.1.4. The likelihood ratio, $f_1(x)/f_0(x)$ takes on the three values $4/9$, $8/9$ and $16/9$, with probabilities $1/4$, $1/2$ and $1/4$ under H_0 and probabilities $1/9$, $4/9$, and $4/9$ under H_1. Rejecting H_0 if $X = 2$ gives the risk point $(1/4, 5/9)$, and rejecting H_0 if $X \ge 1$ gives the point $(3/4, 1/9)$ in the risk set. The complete risk set is given in Figure 2.

5.1.5. The likelihood ratio is $f_1(x)/f_0(x) = e^{x/2}/2$. The best tests reject H_0 when this ratio is greater than some constant, or equivalently, when X is greater than some constant, say $X > c$. The probability of error type I is $\alpha_0 = \mathrm{P}_0(X > c) = e^{-c}$. The probability of error type II is $\alpha_1 = \mathrm{P}_1(X < c) = 1 - e^{-c/2}$. The lower boundary of the risk set therefore satisfies $\alpha_1 = 1 - \sqrt{\alpha_0}$. The complete risk set is given in Figure 3.

Fig. 1.

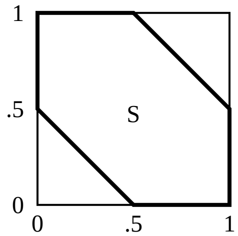


Fig.2.

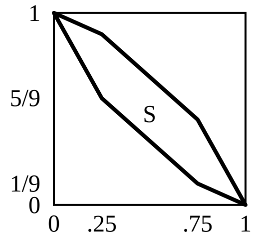
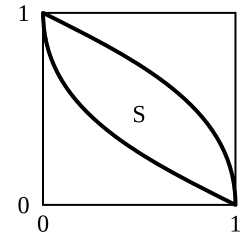


Fig.3.



5.1.6. Since $f(x|\theta) > 0$ for all x, the best tests have the form: $\phi(x)$ is 1, is arbitrary, or is 0 according as the likelihood ratio $\lambda(x) = f_1(x)/f_0(x)$ is greater than k, equal to k, or less than k. In the case where X is $\mathcal{C}(\theta, 1)$ and $\theta_0 = 0$ and $\theta_1 = 1$, the likelihood ratio is

$$\lambda(x) = (1 + x^2)/(1 + (x-1)^2)$$

By evaluating the derivative λ', it may be seen that $\lambda(x)$ starts at 1 at $x = -\infty$, decreases to a minimum at $x = (1-\sqrt{5})/2 = -.618\cdots$, increases to a maximum at $x = (1+\sqrt{5})/2 = 1.618\cdots$, and then decreases to 1 as x tends to ∞. Hence, since $\lambda(1) = \lambda(3) = 2$, the interval $(1, 3)$ is a best test of its size, corresponding to $k = 2$. The power function is $\beta(\theta) = P_\theta(1 < X < 3)$, is symmetric in θ about $\theta = 2$, attains its maximum value of $1/2$ at $\theta = 2$, and decreases to 0 as $\theta \to \infty$. Other values are $\beta(1) = .352\cdots$, $\beta(0) = .148\cdots$, and $\beta(-1) = .070\cdots$.

5.1.7. If ϕ is a best test of size α and $\mathrm{E}_{\theta_1}\phi(X) = \alpha$, then $\phi_1(x) \equiv \alpha$ is also a best test of size α since it has the same power as ϕ. But from the unicity part of the Neyman-Pearson Lemma with $\alpha > 0$, ϕ_1 must have the form (5.7). But since $0 < \alpha < 1$, this implies $f_1(x) = kf_0(x)$ a.s. for some $k \ge 0$. And since both f_1 and f_0 are densities, k must be equal to 1. This implies $\mathrm{P}_{\theta_0} = \mathrm{P}_{\theta_1}$.

5.1.8. The best tests of the form (5.7) become

$$\phi(\mathbf{z}) = \begin{cases} 1 & \text{if } \prod \theta_i'^{z_i} > k \prod \theta_i^{0\,z_i} \\ \gamma(x) & \text{if } \prod \theta_i'^{z_i} = k \prod \theta_i^{0\,z_i} \\ 0 & \text{if } \prod \theta_i'^{z_i} < k \prod \theta_i^{0\,z_i} \end{cases} = \begin{cases} 1 & \text{if } \sum z_i \log r_i > k' \\ \gamma(x) & \text{if } \sum z_i \log r_i = k' \\ 0 & \text{if } \sum z_i \log r_i < k' \end{cases}$$

where $r_i = \theta_i'/\theta_i^0$ and $k' = \log k$. The test of the form (5.8) becomes

$$\phi(\mathbf{z}) = \begin{cases} 1 & \text{if } z_i > 0 \text{ for some } \theta_i^0 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

In the special case $k = 4$, $r_1 = .10/.55$, $r_2 = .40/.20$, $r_3 = .30/.15$, and $r_4 = .20/.10$, the tests reject H_0 for large values of $Z_1 \log(2/11) + (Z_2 + Z_3 + Z_4)\log 2$. But since $Z_1 = n - (Z_2 + Z_3 + Z_4)$, the tests reject H_0 for large values of $Z_2 + Z_3 + Z_4$. Under H_0, $Z_2 + Z_3 + Z_4 \in \mathcal{B}(n, .45)$, and under H_1, $Z_2 + Z_3 + Z_4 \in \mathcal{B}(n, .90)$. The error probabilities may be computed from this.

5.1.9. Note that $\int \phi_0(x) f_0(x)\,dx = \int f_0(x)^+\,dx$. The only functions $\phi(x)$, $0 \le \phi(x) \le 1$, that satisfy $\int \phi(x) f_0(x)\,dx = \int f_0(x)^+\,dx$ are the functions

$$\phi(x) = \begin{cases} 1 & \text{if } f_0(x) > 0 \\ \gamma(x) & \text{if } f_0(x) = 0 \\ 0 & \text{if } f_0(x) < 0 \end{cases}$$

Therefore to maximize $\int \phi(x) f_1(x)\,dx$ out of this class, we may take $\gamma(x)$ to be of the form

$$\gamma(x) = \begin{cases} 1 & \text{if } f_0(x) = 0 \text{ and } f_1(x) > 0 \\ \text{any} & \text{if } f_0(x) = 0 \text{ and } f_1(x) = 0 \\ 0 & \text{if } f_0(x) = 0 \text{ and } f_1(x) < 0 \end{cases}$$

The given $\phi_0(x)$ has $\gamma(x)$ of this form.