Solutions to the Exercises of Section 2.9.

2.9.1. Let δ be any randomized decision rule. Then δ is a distribution on $\mathcal{A} = \{1, 2, \ldots\}$. Let $\epsilon > 0$ be arbitrary. One can find an integer M such that $\delta(\{M, M+1, \ldots\}) < \epsilon$. Then

$$\sup_\tau r(\tau, \delta) = \sup_\theta r(\theta, \delta) \geq r(M, \delta) \geq (1-\epsilon) - \epsilon = 1 - 2\epsilon.$$

Since this holds for all $\epsilon > 0$, we have $\sup_\tau r(\tau, \delta) = +1$, giving (2.23). Equation (2.24) follows by symmetry. Any rule δ for the statistician guarantees a loss no greater than $\bar{V} = +1$, so every rule is minimax for the statistician.

2.9.2. Let the risk set be the square, $S = \{(x, y) : 1 \leq x \leq 2, 0 \leq y \leq 1\}$. Then the rule δ_0 with risk point $(x, y) = (1, 0)$ is minimax (and all other risk points on the line joining $(1, 0)$ to $(1, 1)$ are minimax as well), and Bayes with respect to every prior distribution. But τ_0 giving probablity one to the first coordinate is the only least favorable distribution.

2.9.3. First, suppose that $f(x)$ is lower semicontinuous and that x_0 is in the domain of f. We are to show $\liminf_{x \to x_0} f(x) \geq f(x_0)$, or equivalently,

$(*)$ for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - x_0| < \delta$ implies $f(x) > f(x_0) - \epsilon$.

Let $c = f(x_0)$, $\epsilon > 0$ and $S = \{x : f(x) > c - \epsilon\}$. Then $x_0 \in S$ and the lower semicontinuity of f imply that S is open. Then there is a neighborhood of x_0 that is contained in S as well. That is, there exists a $\delta > 0$ such that $\{x : |x - x_0| < \delta\} \subset S$. This iimplies $(*)$.

To show the converse, suppose $(*)$ holds for all x_0 in the domain of f and let c be arbitrary. We are to show that $S = \{x : f(x) > c\}$ is open. Suppose $x_0 \in S$. Then $f(x_0) > c$, so that $(*)$ with $\epsilon = f(x_0) - c$ implies that there is a $\delta > 0$ such that $|x - x_0| < \delta$ implies that $f(x) > f(x_0) - \epsilon = c$. Thus there is a neighborhood about x_0 that is contained in S showing that S is open.

2.9.4. Suppose $f_\theta(x)$ is a lower semicontinuous function of x for all $\theta \in \Theta$, and let $g(x) = \sup_{\theta \in \Theta} f_\theta(x)$. For any real number c,

$$\{x : g(x) > c\} = \{x : \sup_{\theta \in \Theta} f_\theta(x) < c\} = \bigcup_{\theta \in \Theta} \{x : f_\theta(x) < c\}.$$

From the definition of lower semicontinuity, each of the sets $\{x : f_\theta(x) < c\}$ is open, and since the union of an arbitrary number of open sets is open, we have that $\{x : g(x) > c\}$ is open, showing that $g(x)$ is lower semicontinuous.

2.9.5. If D is finite, say $D = \{d_1, \ldots, d_n\}$, then D^* is may be taken as the probability simplex, $D^* = \{\mathbf{p} = (p_1, \ldots, p_n) : p_1 \geq 0, \cdots, p_n \geq 0, \sum_1^n p_i = 1\}$. This is compact in the Euclidean topology, and the risk function, $R(\theta, \mathbf{p}) = \sum_1^n p_i R(\theta, d_i)$, is a continuous function of $\mathbf{p} \in \mathbf{D}^*$. Therefore, we may take $C = D^*$ in Theorem 2, since it is essentially complete, compact, and R is continuous. We conclude the game has a value and the statistician has a minimax strqtegy.

2.9.6. The nonrandomized risk set, S_0, is in Euclidean k-dimensional space and S is the convex hull of S_0 (Corollary 2.7.1). By Lemma 2.4.1, every point of S is a mixture of at most $k+1$ points of S_0. Thus <u>every</u> randomized decision rule is equivalent to one giving mass to at most $k+1$ nonrandomized rules. This is then true of the minimax rule for the statistician which exists in this problem because the risk set is bounded from below and closed from below.