Solutions to the Exercises of Section 2.7.

2.7.1. Let S_1 be a convex set in E_k, let $\mathcal{A} = S_1$, let $\Theta = \{\theta_1, \ldots, \theta_k\}$ and consider the game $(\Theta, \mathcal{A}, L)$ with $L(\theta_j, \mathbf{a}) = a_j$. If the random variable available to the statistician is degenerate at 0 for all $\theta \in \Theta$, then $D = \mathcal{A}$, $D^* = \mathcal{A}^*$, and (Θ, D^*, R) is the same as $(\Theta, \mathcal{A}^*, L)$. The risk set of Equation (2.3) reduces to

$$S = \{(y_1, \ldots, y_k) : \text{for some } \delta \in \mathcal{A}^*,\ y_j = L(\theta_j, \delta),\ \text{for } j = 1, \ldots, k\}.$$

We are to show $S = S_1$.
(a) $S_1 \subset S$: Let $\mathbf{a} \in S_1$ and let δ be degenerate at $\mathbf{a}$. Then $a_j = L(\theta_j, \delta)$ so $\mathbf{a} \in S$.
(b) $S \subset S_1$: Let $\mathbf{y} \in S$ and find a distribution δ over $\mathcal{A}$ such that $y_j = L(\theta_j, \delta)$ for all j. Then $y_j = \mathrm{E}L(\theta_j, \mathbf{Z}) = \mathrm{E}Z_j$ where $\mathbf{Z}$ has distribution δ. Since S_1 is convex, we have $\mathbf{y} = \mathrm{E}\mathbf{Z} \in S_1$ from Lemma 3, thus showing $S \subset S_1$.

2.7.2. Suppose $\Theta = \{\theta_1, \ldots, \theta_k\}$ is finite, D is compact, and $R(\theta, d)$ is continuous in d for each $\theta \in \Theta$. Then, the nonrandomized risk set, S_0, is the continuous image of the compact set, D, and hence is compact. Since S is the convex hull of S_0 by Corollary 1, and since the convex hull of a compact set is compact by Theorem 2.4.2, it follows that S is compact.

2.7.3. Let S_1 and S_2 be disjoint closed convex subsets of k-space, and suppose that S_1 is bounded and hence compact. Let $S = \{\mathbf{z} : \mathbf{z} = \mathbf{x} - \mathbf{y}\}$. Then S is convex and $\mathbf{0} \notin S$ as in the proof of Theorem 1. Moreover, S is closed. (**Proof**. If $\mathbf{z}_n \in S$ and $\mathbf{z}_n \to \mathbf{z}$, find $\mathbf{x}_n \in S_1$ and $\mathbf{y}_n \in S_2$ such that $\mathbf{z}_n = \mathbf{x}_n - \mathbf{y}_n$. Since S_1 is compact, there exists a subsequence $\mathbf{x}_{n'}$ that converges, say $\mathbf{x}_{n'} \to \mathbf{x} \in S_1$. Then, $\mathbf{y}_{n'} = \mathbf{x}_{n'} - \mathbf{z}_{n'} \to \mathbf{x} - \mathbf{z} = \mathbf{y} \in S_2$, so $\mathbf{z} = \mathbf{x} - \mathbf{y} \in S$. $\blacksquare$) Now by Lemma 1, there is a $\mathbf{p}$ such that $\mathbf{p}^T\mathbf{z} > 0$ for all $\mathbf{z} \in S$. Since S is closed, $\epsilon = \inf_{\mathbf{z} \in S} \mathbf{p}^T\mathbf{z} > 0$, which implies $0 < \epsilon = \inf_{\mathbf{x} \in S_1, \mathbf{y} \in S_2} \mathbf{p}^T(\mathbf{x} - \mathbf{y}) = \inf_{\mathbf{x} \in S_1} \mathbf{p}^T\mathbf{x} - \sup_{\mathbf{y} \in S_2} \mathbf{p}^T\mathbf{y}$, completing the proof.

2.7.4. In two dimensions, let $S_1 = \{(x_1, x_2) : x_1 > 0, x_2 \geq 1/x_1\}$ and $S_2 = \{(y_1, y_2) : y_1 = 0, -\infty < y_2 < \infty\}$. Then S_1 and S_2 are disjoint closed and convex sets. The separating hyperplane is unique and is given by $\mathbf{p}^T = (1, 0)$. Yet, $\inf_{\mathbf{x} \in S_1} \mathbf{p}^T\mathbf{x} = 0$ and $\sup_{\mathbf{y} \in S_2} \mathbf{p}^T\mathbf{y} = 0$.

2.7.5. Suppose S is strictly convex and $\mathbf{x}_0$ is not an interior point of S. If $\mathbf{x}_0$ is not on the boundary of S, then $\underset{\sim}{x}_0$ is not in the closure which is also convex so by Lemma 1 there is a $\mathbf{p} \neq \mathbf{0}$ such that $\mathbf{p}^T(\mathbf{x} - \mathbf{x}_0) > 0$ for all $\mathbf{x} \in S$ and we are done. So assume that $\mathbf{x}_0$ is on the boundary of S. By Theorem 1, there exists a point $\mathbf{p} \neq \mathbf{0}$ such that $\mathbf{p}^T\mathbf{x} \geq \mathbf{p}^T\mathbf{x}_0$ for all $\mathbf{x} \in S$.

Suppose $\mathbf{p}^T\mathbf{x} = \mathbf{p}^T\mathbf{x}_0$ for some $\mathbf{x} \in S$, $\mathbf{x} \neq \mathbf{x}_0$. If $\mathbf{x}$ is on the boundary of S, then since S is strictly convex, the point $(\mathbf{x} + \mathbf{x}_0)/2$ is in the interior of S and $\mathbf{p}^T(\mathbf{x} + \mathbf{x}_0)/2 = \mathbf{p}^T\mathbf{x}_0$. Thus we may assume without loss of generality that $\mathbf{x}$ is in the interior of S. But then $\mathbf{y} - \epsilon\,\mathbf{p}$ is in the interior of S for sufficiently small ϵ, and this implies that $\mathbf{p}^T\mathbf{x}_0 \leq \mathbf{p}^T(\mathbf{x} - \epsilon\,\mathbf{p}) = \mathbf{p}^T\mathbf{x} - \epsilon\,\mathbf{p}^T\mathbf{p} < \mathbf{p}^T\mathbf{x} = \mathbf{p}^T\mathbf{x}_0$, a contradiction that completes the proof.