Solutions to the Exercises of Section 2.5.

2.5.1. Let S be convex set in E_k, and let $\overline{S}$ denote its closure. Let $x \in \overline{S}$ and $y \in \overline{S}$, and let $0 \le \lambda \le 1$. We are to show that $\lambda x + (1-\lambda)y \in \overline{S}$. Since $x \in \overline{S}$, we can find a sequence $x_n \in S$ such that $x_n \to x$ as $n \to \infty$. Similarly, we can find $y_n \in S$ such that $y_n \to y$ as $n \to \infty$. Since S is convex, $\lambda x_n + (1-\lambda)y_n \in S$. Then, since $\lambda x_n + (1-\lambda)y_n \to \lambda x + (1-\lambda)y$, we have $\lambda x + (1-\lambda)y \in \overline{S}$.

2.5.2. Let S be closed from below and bounded, and let $\mathbf{p}^T = (p_1, \ldots, p_k)$ be an arbitrary prior distribution. Let $\alpha_0 = \inf\{\alpha : \mathbf{p}^T\mathbf{x} \le \alpha \quad \text{for some} \quad x \in S\}$. First we find $\mathbf{x}_n \in S$ such that $\mathbf{p}^T\mathbf{x}_n \le \alpha_0 + (1/n)$. Then since S is bounded, we may find a subsequence of the $\mathbf{x}_n$ that converges to some $\mathbf{x}$. This $\mathbf{x}$ must be in $\overline{S}$. Although this point has minimum Bayes risk, $\mathbf{p}^T\mathbf{x} = \alpha_0$, $\mathbf{x}$ might not be in S. However, consider the set $S' = Q_\mathbf{x} \cap \overline{S}$. All points $\mathbf{y} \in S'$ have the same Bayes risk. (The Bayes risk cannot be any better since $\mathbf{y} \in \overline{S}$, and it cannot be any worse since $\mathbf{y} \in Q_\mathbf{x}$.) Moreover, S' is closed (the intersection of two closed sets), convex (the intersection of two convex sets), and nonempty ($\mathbf{x} \in S'$). Hence, by Lemma 1, the lower boundary of S' is not empty, $\lambda(Q_\mathbf{x} \cap \overline{S}) \ne \emptyset$. However, the lower boundary of S' is contained in the lower boundary of S, because if $\mathbf{y} \in \lambda(Q_\mathbf{x} \cap \overline{S})$, then $\{\mathbf{y}\} = Q_\mathbf{y} \cap \overline{(Q_\mathbf{x} \cap \overline{S})} = Q_\mathbf{y} \cap Q_\mathbf{x} \cap \overline{S} = Q_\mathbf{y} \cap \overline{S}$. Since S is closed from below, all points of $\lambda(Q_\mathbf{x} \cap \overline{S})$ are in S and all have the minimum Bayes risk.

2.5.3. We are given that the risk set, S, is bounded from below and closed from below and that δ_0 is admissible. Let $\mathbf{x} = (R(\theta_1, \delta_0), \ldots, R(\theta_k, \delta_0))$. We are to show that $\mathbf{x} \in \lambda(S)$.

Let $S_1 = Q_\mathbf{x} \cap \overline{S}$. Then S_1 is nonempty ($\mathbf{x} \in S_1$), convex (both $Q_\mathbf{x}$ and $\overline{S}$ are convex), and bounded from below (since S is bounded from below). Lemma 2.5.1 applies to give us that $\lambda(S_1)$ is not empty. Let $\mathbf{y} \in \lambda(S_1)$. Then using the fact that S_1 is closed,

$$\{\mathbf{y}\} = Q_\mathbf{y} \cap \overline{S}_1 = Q_\mathbf{y} \cap S_1 = Q_\mathbf{y} \cap Q_\mathbf{x} \cap \overline{S} = Q_\mathbf{y} \cap \overline{S},$$

since $\mathbf{y} \in Q_\mathbf{x}$ implies that $Q_\mathbf{y} \subseteq Q_\mathbf{x}$. This shows that $\mathbf{y} \in \lambda(S)$ and since S is closed from below, $\mathbf{y} \in S$. But since δ_0 is admissible, $\mathbf{x}$ is the only point in $Q_\mathbf{x} \cap S$. That is, $\mathbf{x} = \mathbf{y} \in \lambda(S)$.

2.5.4. (a) The counterexample of Exercise 2.4.4 is also a counterexample here.

(b) Let $S = \{(x, y) : x \le 0, 0 < y \le 1\} \cup \{(0,0)\}$. Then since $\lambda(S) = \emptyset \subseteq S$, S is closed from below. But $\{(0,0)\} \notin \lambda(S)$, although the δ corresponding to $(0,0)$ is admissible.