Solutions to the Exercises of Section 2.4.

2.4.1. Suppose S is closed and convex, and let $\mathbf{x} \in \lambda(S)$. Then $Q_{\mathbf{x}} \cap \overline{S} = \{\mathbf{x}\}$. This shows that $\mathbf{x} \in \overline{S}$. But since S is closed $S = \overline{S}$ so that $\mathbf{x} \in S$. Thus, $\lambda(S) \subset S$, so S is closed from below.

2.4.2. Suppose $\mathbf{x} = (R(\theta_1, \delta_0), \ldots, R(\theta_k, \delta_0)) \in \lambda(S)$ where S is the risk set. Then $Q_{\mathbf{x}} \cap \overline{S} = \{\mathbf{x}\}$. If δ_0 were not admissible, then there is a rule δ_1 better than δ_0; that is, $\mathbf{y} = (R(\theta_1, \delta_1), \ldots, R(\theta_k, \delta_1)) \in Q_{\mathbf{x}}$ and $\mathbf{y} \neq \mathbf{x}$. Since $\mathbf{y} \in S$, we have $\mathbf{y} \in Q_{\mathbf{x}} \cap \overline{S}$, contradicting $Q_{\mathbf{x}} \cap \overline{S} = \{\mathbf{x}\}$.

2.4.3. Let $\mathbf{x} = (R(\theta_1, \delta_0), \ldots, R(\theta_k, \delta_0))$. Then δ_0 is admissible if $Q_{\mathbf{x}}$ contains no other points of S, i.e. if $Q_{\mathbf{x}} \cap S = \{\mathbf{x}\}$. If in addition S is closed, then $S = \overline{S}$, so that $Q_{\mathbf{x}} \cap \overline{S} = \{\mathbf{x}\}$, which by definition of the lower boundary says that $x \in \lambda(S)$

2.4.4. Let the risk set be $S = \{(x, y) : 0 < x \leq 1, 0 < y \leq 1\} \cup \{(0, 1)\}$. Then any δ with risk point $(0, 1)$ is admissible. But the only lower boundary point of S is $(0, 0)$.

2.4.5. Let $\mathbf{x} \in \lambda(S)$. We must show that $\mathbf{x} \in \partial S$; that is, we must show that $\mathbf{x} \in \overline{S}$ and $\mathbf{x} \notin S^o$. $\mathbf{x} \in \lambda(S)$ means that $Q_{\mathbf{x}} \cap \overline{S} = \{\mathbf{x}\}$, so one immediately has $\mathbf{x} \in \overline{S}$. To show $\mathbf{x} \notin S^o$, we must show that no open set containing X is contained in S. But any open set containing $\mathbf{x}$ contains points of $Q_{\mathbf{x}}$ other than X itself, and these are not in S.