Solutions to the Exercises of Section 1.4.

1.4.1. **Proof.** (i) From linearity of $\leq$, either $p \leq p$ or $p \leq p$. Thus, $p \leq p$ and $p \sim p$.
(ii) If $p_1 \leq p_2$ and $p_2 \leq p_1$, then $p_2 \leq p_1$ and $p_1 \leq p_2$. So $p_1 \sim p_2$ implies $p_2 \sim p_1$.
(iii) If $p_1 \leq p_2$ and $p_2 \leq p_1$, and if $p_2 \leq p_3$ and $p_3 \leq p_2$, then from the transitivity of $\leq$, we have $p_1 \leq p_3$ and $p_3 \leq p_1$, so that $p_1 \sim p_3$.

1.4.2. **Proof.** We are given $p_1 \leq p_2$ and $p_2 \leq p_3$ but not $p_3 \leq p_2$. Then $p_1 \leq p_3$ by transitivity. We must show not $p_3 \leq p_1$. If $p_3 \leq p_1$, then by transitivity $p_3 \leq p_2$. This contradicts not $p_3 \leq p_2$, completing the proof.

1.4.3. The statement is not quite correct. We should replace H_1 with the following:
H_1': Suppose p_n and p_n' are in $\mathcal{P}^*$ for $n = 1, 2, \ldots$, and $\lambda_n \geq 0$ and $\sum_1^\infty \lambda_n = 1$. If $p_n \leq p_n'$ for all n, then $\sum_{n=1}^\infty \lambda_n p_n \leq \sum_{n=1}^\infty \lambda_n p_n'$. If, in addition, $p_n < p_n'$ for some n for which $\lambda_n > 0$, then $\sum_{n=1}^\infty \lambda_n p_n < \sum_{n=1}^\infty \lambda_n p_n'$.

We may then state the theorem as follows.

Theorem. *If a preference pattern $\leq$ on $\mathcal{P}^*$ satisfies H_1' and H_2, then there exists a utility, u, on $\mathcal{P}^*$ which agrees with $\leq$. Furthermore, u is uniquely determined up to a linear transformation. Moreover, u is bounded, and*

$$(*) \qquad u\Big(\sum_{n=1}^\infty \lambda_n p_n\Big) = \sum_{n=1}^\infty \lambda_n u(p_n).$$

Proof. (Blackwell and Girshick) Since H_1' implies H_1, the first two statements of the theorem folow from Theorem 1 of the text. Now suppose that u is not bounded. Assume without loss of generality that it is not bounded from above. Then we can find a sequence $p_n \in \mathcal{P}^*$ such that $u(p_n) > 2^n$ and $u(p_n) > u(p_{n-1})$ for all n. Let $q = \sum_1^\infty 2^{-n} p_n$ and $q_N = (\sum_1^N 2^{-n} p_n) + 2^{-N} p_N$. Now hypothesis H_1' implies that $q_N < q$ for all N, so that $u(q_N) < u(q)$ for all N. But q_N is a finite mixture so we may compute $u(q_N) > N+1$. This implies that $u(q) > N+1$ for all N which contradicts that requirement that $u(q)$ be finite and shows that u is bounded.

Now note that $(*)$ is automatically true if λ_n is zero except for a finite number of values of n. So assume that $\lambda_n > 0$ for an infinite number of values of n so that $\sum_{N+1}^\infty \lambda_n > 0$ for all N. Then

$$\begin{aligned} u\Big(\sum_1^\infty \lambda_n p_n\Big) &= u\left[\sum_1^N \lambda_n p_n + \Big(\sum_{N+1}^\infty \lambda_n\Big) \sum_{N+1}^\infty \mu_n^{(N)} p_n\right] \\ &= \sum_1^N \lambda_n u(p_n) + \sum_{N+1}^\infty \lambda_n u\Big(\sum_{N+1}^\infty \mu_n^{(N)} p_n\Big) \end{aligned}$$

where $\mu_n^{(N)} = \lambda_n / \sum_{N+1}^\infty \lambda_n$. Since u is bounded,

$$\sum_{N+1}^\infty \lambda_n u\Big(\sum_{N+1}^\infty \mu_n^{(N)} p_n\Big) \to 0$$

as $N \to \infty$, completing the proof of $(*)$.

1.4.4. Let $0 < \lambda \leq 1$. By the definition of $\sim$, $p_1 \sim p_2$ is equivalent to $p_1 \leq p_2$ and $p_2 \leq p_1$. From hypothesis H_1 applied twice, this is equivalent to $\lambda p_1 + (1-\lambda)q \leq \lambda p_2 + (1-\lambda)q$ and $\lambda p_2 + (1-\lambda)q \leq \lambda p_1 + (1-\lambda)q$. Again by the definition of $\sim$, this is equivalent to $\lambda p_1 + (1-\lambda)q \sim \lambda p_2 + (1-\lambda)q$.

1.4.5. Suppose that $\pi_1, \ldots, \pi_m$ and $\pi_1', \ldots, \pi_m'$ are two probability vectors such that

$$u_g[p_1, \ldots, p_m] = u(p_1)\pi_1 + \cdots + u(p_m)\pi_m = u(p_1)\pi_1' + \cdots + u(p_m)\pi_m' \quad \text{for all} \quad p_1, \ldots, p_m \in \mathcal{P}^*.$$

Find $q_0 < q_1$ so that $u(q_1) > u(q_0)$. Now for fixed i take $p_i = q_1$ and $p_j = q_0$ for $j \neq i$ in this equation. We find $u(q_1)\pi_i + \sum_{j \neq i} u(q_0)\pi_j = u(q_1)\pi_i' + \sum_{j \neq i} u(q_0)\pi_j'$. This reduces to $\pi_i(u(q_1) - u(q_0)) + u(q_0) = \pi_i'(u(q_1) - u(q_0)) + u(q_0)$. But since $u(q_1) > u(q_0)$, this implies that $\pi_i = \pi_i'$. Since i is arbitrary, this shows uniqueness.