Large Sample Theory
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Exercises, Section 23, Minimum Chi-Square Estimates.

1. Let $X_1, \ldots, X_n$ be a sample from the $\mathcal{G}(1, \theta)$ distribution, with density $f(x) = \theta^{-1} e^{-x/\theta} \mathrm{I}(x > 0)$, and let $Y_1, \ldots, Y_n$ be an independent sample from the $\mathcal{G}(1, \theta^2)$ distribution, where $\theta > 0$ is an unknown parameter.

(a) Find the χ^2, $Q_n(\pi(\theta))$ of Example 1, where $Z_n = (\overline{X}_n, \overline{Y}_n)^T$.

(b) Find the transformed χ^2 with the transformation $g(x, y) = (x, \sqrt{y})$.

(c) Find the minimum modified, transformed χ^2 estimate of θ. (Ans. $\tilde{\theta}_n = (\overline{X}_n + 4\sqrt{\overline{Y}_n})/5$.) Compare to the maximum likelihood estimate. Which is better asymptotically?

2. Let $\boldsymbol{X} = (X_1, \ldots, X_c)$ have a multinomial distribution with sample size $n = 1$ and probability vector $\boldsymbol{p}(\boldsymbol{\theta}) = (p_1(\boldsymbol{\theta}), \ldots, p_c(\boldsymbol{\theta}))^T > 0$, where $\mathbf{1}^T \boldsymbol{p}(\boldsymbol{\theta}) = \sum_1^c p_i(\boldsymbol{\theta}) = 1$ for all $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)^T \in \Theta$, an open set in k-dimentions, $k < c$. Assume $\dot{\boldsymbol{p}}(\boldsymbol{\theta})$ (a c by k matrix) exists for all $\boldsymbol{\theta} \in \Theta$, and note that $\mathbf{1}^T \dot{\boldsymbol{p}}(\boldsymbol{\theta}) = \mathbf{0}$. Show that Fisher Information is $\mathcal{I}(\boldsymbol{\theta}) = \dot{\boldsymbol{p}}(\boldsymbol{\theta})^T \boldsymbol{P}(\boldsymbol{\theta})^{-1} \dot{\boldsymbol{p}}(\boldsymbol{\theta})$, where $\boldsymbol{P}(\boldsymbol{\theta}) = \mathrm{diag}(\boldsymbol{p}(\boldsymbol{\theta}))$.