Large Sample Theory
Ferguson

Exercises, Section 22, Asymptotic Distribution of the Likelihood Ratio Test Statistic.

1. Consider the model, $Y_i = \alpha + \beta x_i + \epsilon_i$ for $i = 1, 2, \ldots, n$, where the x_i are known numbers not all equal, where the ϵ_i are i.i.d. $\mathcal{N}(0, \sigma^2)$, and where the parameters $(\alpha, \beta, \sigma^2)$ are unknown.

(a) What is the likelihood ratio test of the hypothesis $H_0 : \alpha = \beta$? Give its exact distribution.

(b) What is the likelihood ratio test of H_0 when σ^2 is known? Give its exact distribution.

2. (a) Let X_{ij} be independent random variables with $X_{ij} \in \mathcal{P}(\lambda_{ij})$ for $i = 1, \ldots, n$, and $j = 1, \ldots, k$. Find the likelihood ratio test of $H_0 : \lambda_{ij} = j\lambda$ for some $\lambda > 0$, for $i = 1, \ldots, n$, and $j = 1, \ldots, k$, against $H_1 - H_0$ where $H_1 : \lambda_{ij}$ is independent of i (i.e. $\lambda_{ij} = \lambda_j$ for some numbers $\lambda_j > 0$), $i = 1, \ldots, n$, and $j = 1, \ldots, k$.

(b) Describe the asymptotic distribution of the likelihood ratio test statistic under H_0 as $n \to \infty$, with k fixed.

3. Let X_{ij} be independent with X_{ij} having the exponential distribution with density $f(x|\beta_i) = \beta_i^{-1} e^{-x/\beta_i}$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n$.

(a) Find the likelihood ratio test of the hypothesis $H_0 : \beta_1 = \beta_2 = \cdots = \beta_k$.

(b) Describe how to find the cutoff point for a size α test of H_0 when n is large.

4. (a) Let $X_1, \ldots, X_n$ be a sample from $\mathcal{N}(\theta_1, 1)$, and let $Y_1, \ldots, Y_n$ be an independent sample from $\mathcal{N}(\theta_2, 1)$. Find the likelihood ratio test of the hypothesis $H_0 : \theta_1 = \theta_2 = 0$, within the general hypothesis, $H : \theta_1 \geq 0$, or $\theta_2 \geq 0$. (The parameter space is the plane with the negative quadrant removed.)

(b) What is the (asymptotic) distribution of $-2 \log \lambda_n$ for this problem? You should be able to give directions for finding the cutoff point for a given level of significance, α.

5. Let $X_1, \ldots, X_n$ be a sample from the Poisson distribution with density $f(x|\mu) = e^{-\mu}\mu^x/x!$ for $x = 0, 1, \ldots$, and let $Y_1, \ldots, Y_n$ be an independent sample from the Poisson distribution with density $f(y|\theta)$, where $\mu > 0$ and $\theta > 0$ are unknown parameters.
(a) Find the likelihood ratio test statistic for testing $H_0 : \mu = \theta^2$.
(b) What is its asymptotic distribution under H_0?

6. Under H_1, $X_1, \ldots, X_n$ are independent with $X_i \in \mathcal{G}(1, \theta_i)$ (exponential distribution with mean θ_i. Under H_0, all θ_i are equal, say $X_i \in \mathcal{G}(1, \theta)$.

(a) Find the likelihood ratio test statistic, λ_n, for testing H_0 against H_1.

(b) Find the asymptotic distribution of $\log \lambda_n$, suitably normalized, under H_0.

(c) Why doesn't Theorem 22 apply?