Large Sample Theory
Ferguson

Exercises, Section 21, Asymptotic Normality of Posterior Distributions.

1. Let $X_1, \ldots, X_n$ be a sample from the exponential distribution with density

$$f(x|\theta) = \theta e^{-\theta x} \qquad \text{for } x > 0$$

for some $\theta > 0$. Let the prior distribution of θ be the gamma distribution with density

$$g(\theta|\alpha, \eta) = \frac{\eta^\alpha}{\Gamma(\alpha)} e^{-\eta\theta}\theta^{\alpha-1} \qquad \text{for } \theta > 0$$

where η and α are given positive numbers.

(a) Find the posterior distribution of θ given $X_1, \ldots, X_n$.

(b) For the loss function,

$$L(\theta, a) = \begin{cases} 2(\theta - a) & \text{if } a < \theta \\ a - \theta & \text{if } a \geq \theta \end{cases}$$

(underestimation is the more serious error), what is the Bayes estimate? Suppose $\alpha = 4$, $\eta = 2$, $\overline{X}_n = 1$ and $n = 6$ or $n = 46$. Find the Bayes estimate (for both values of n) using the electronic statistical tables on the web.

(c) For the same values of α, η, and $\overline{X}_n$, find the approximate Bayes estimate using the approximation to the posterior distribution of θ given by the Bernstein-von Mises Theorem.

2. Let $X_1, \ldots, X_n$ be a sample from the geometric distribution with probability mass function, $f(x|\theta) = (1-\theta)\theta^x$ for $x = 0, 1, \ldots$, for some $\theta \in (0, 1)$. Suppose the prior density of θ on (0,1) is $g(\theta) = (\pi/2)\sin(\pi\theta)$. Find an approximation to the posterior density of θ that is good for sufficiently large samples.

3. A coin with probability θ of heads is tossed n times and the number of heads, X_n, is observed. The prior distribution of θ is uniform on the interval (0,1).

(a) Find the posterior density of θ given X_n.

(b) What is the approximation to this density given by the Bernstein-von Mises Theorem?