Large Sample Theory
Ferguson

Exercises, Section 19, The Cramér-Rao Lower Bound.

1. (Relation of Kullback-Leibler information to Fisher information.) Let $f_\theta(x)$ be the density of a one-parameter exponential family of distributions in natural parametrization, $f_\theta(x) = \exp\{\theta T(x) - c(\theta)\}h(x)$. Let θ_0 be an interior point of the natural parameter space. Find the Kullback-Leibler information number, $K(f_{\theta_0}, f_\theta)$, for an arbitrary θ and show that as $\theta \to \theta_0$, $K(f_{\theta_0}, f_\theta) \sim (\theta-\theta_0)^2\mathcal{I}(\theta_0)/2$, where $\mathcal{I}(\theta)$ is Fisher information.

2. (a) Prove the following version of the Information Inequality that is valid without assuming derivatives. Assume that the support of $f(x|\theta)$ does not depend on θ, and let $\hat\theta(x)$ be a statistic with finite expectation, $g(\theta) = E_\theta(\hat\theta)$, for all θ. Then for all θ' and θ,

$$\mathrm{Var}_\theta(\hat\theta(X)) \geq \frac{(g(\theta') - g(\theta))^2}{\mathrm{Var}_\theta(f(X|\theta')/f(X|\theta))}.$$

(b) Dividing the numerator and denominator on the right by $(\theta' - \theta)^2$ and letting $\theta' \to \theta$ gives the usual Information Inequality. This shows that we can define Fisher information more generally by $\mathcal{I}(\theta) = \lim_{\theta'\to\theta}(\theta'-\theta)^{-2}\mathrm{Var}_\theta(f(X|\theta')/f(X|\theta))$. Using this definition, find Fisher information for the double exponential distribution, $f(x|\theta) = (1/2)\exp\{-|x-\theta|\}$. Does the maximum likelihood estimate achieve the Cramér-Rao bound in the limit as the sample size goes to infinity?

(c) Find Fisher information for density $f(x|\theta) = \theta\,\mathrm{I}(0 < x < \theta) + (1+\theta)\,\mathrm{I}(\theta < x < 1)$, where the parameter space is $\Theta = (0, 1)$. Can you find an estimate of θ asymptotically as good as is indicated by Fisher information?

3. (a) Find Fisher information for the two-parameter extreme-value distribution with distribution function $F(x|\mu,\sigma) = \exp\{-e^{-(x-\mu)/\sigma}\}$. Using Maple, one finds that $-\int_0^\infty \log(y)e^{-y}\,dy = \gamma =$ Euler's constant $= .57722\cdots = -\Gamma'(1)$.

(b) Find the Cramér-Rao lower bound to the variance of an unbiased estimate of μ/σ based on a sample of size n.

4. (Sequential Information Inequality.) Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables whose density, $f(x|\theta)$, exists and satisfies the conditions of Theorem 19. Let N be a stopping time (i.e. N is a random variable taking positive integer values such that for all n, the event $\{N = n\}$ depends only on the variables $X_1, \ldots, X_n$). Let $\{\hat\theta_n(X_1,\ldots,X_n\}_{n=1}^\infty$ be a sequence of estimates of θ with finite expectations and consider the estimate $\hat\theta_N = \sum_1^\infty \hat\theta_n(X_1,\ldots,X_n)\,I\{N = n\}$. Let $g(\theta) = E_\theta\hat\theta_N$. Show that

$$\mathrm{Var}_\theta(\hat\theta_N) \geq \frac{g'(\theta)^2}{\mathcal{I}(\theta)E_\theta N}.$$

(You may use Wald's equations: If $Y_1, Y_2, \ldots$ are i.i.d. with finite mean, μ, and if N is a stopping time depending on the Y's such that $\mathrm{E}N < \infty$, then $\mathrm{E}(Y_1 + \cdots + Y_N) = \mu\mathrm{E}(N)$. If in addition $\mu = 0$ and the variance is finite, then $\mathrm{E}(Y_1 + \cdots + Y_N)^2 = \mathrm{Var}(Y_1)\mathrm{E}(N)$.)

5. Consider the linear regression model with observations $Y_i = \alpha + \beta x_i + \epsilon_i$ for $i = 1, \ldots, n$, where the x_i are known constants, α and β are unknown parameters, and ϵ_i are independent normal random errors with mean zero and known variance, σ^2.
(a) Find the Fisher information matrix, $\mathcal{I}_n(\alpha, \beta)$ for this problem.
(b) Using the Fisher information matrix, find the Cramér-Rao lower bound for the variance of an unbiased estimate of β. Is it attained?
(c) What is the Cramér-Rao lower bound for the variance of an unbiased estimate of β when α is known? Is it attained?