Large Sample Theory
Ferguson

Exercises, Section 6, Slutsky Theorems.

1. (G. Blom) An urn contains one white and one black ball. Draw a ball at random. With probability 1/2, return it to the urn; otherwise (again with probability 1/2) put a ball of the opposite color in the urn. Perform n such drawings in succession. Find the limiting distribution of $(X_n - \mathrm{E}X_n)/\sqrt{n}$, where X_n is the number of white balls appearing in the n draws.

2. Let $X_1, X_2, \ldots$ be i.i.d. double exponential (Laplace) random variables with density, $f(x) = (2\tau)^{-1}\exp\{-|x|/\tau\}$, where τ is a positive parameter that represents the mean deviation, $\tau = \mathrm{E}|X|$. Let $\overline{X}_n = n^{-1}\sum_1^n X_i$ and $\overline{Y}_n = n^{-1}\sum_1^n |X_i|$.
(a) Find the joint asymptotic distribution of $\overline{X}_n$ and $\overline{Y}_n$.
(b) Find the asymptotic distribution of $(\overline{Y}_n - \tau)/\overline{X}_n$.

3. Suppose

$$\sqrt{n}\left(\begin{pmatrix} X_n \\ Y_n \\ Z_n \end{pmatrix} - \begin{pmatrix} \mu_x \\ \mu_y \\ \mu_z \end{pmatrix}\right) \xrightarrow{\mathcal{L}} \begin{pmatrix} U \\ V \\ W \end{pmatrix} \in \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_z^2 \end{pmatrix}\right)$$

Show using Slutsky's Theorem that

$$\sqrt{n}(X_n + Y_n Z_n - \mu_x - \mu_y\mu_z) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2)$$

for some σ^2, and find σ^2.

4. Suppose $X_1, \ldots, X_n$ is a sample from the uniform distribution on the interval $(0, \theta)$, $\mathcal{U}(0, \theta)$. The maximum likelihood estimate of θ is M_n, the maximum of the sample. In Chapter 14, we will see that $n(\theta - M_n) \xrightarrow{\mathcal{L}} Z$, where Z has the exponential distribution, $\mathcal{G}(1, \theta)$. As an estimate of θ, M_n might not be so good since $M_n < \theta$ with probability 1, but we might use $((n + c)/n)M_n$ for some positive number c.
(a) What is the asymptotic distribution of $((n + c)/n)M_n$?
(b) What value of c should be used if we measure the accuracy of the estimate by squared error loss?
(c) What value of c should be used if we measure the accuracy of the estimate by absolute error loss?

5. Suppose X_n has a binomial distribution with sample size, n, and probability of success, p. Let $Y_n = \max\{\frac{X_n}{n}, 1 - \frac{X_n}{n}\}$. What is the asymptotic distribution of Y_n,
(a) when $p \neq 1/2$?
(b) when $p = 1/2$?

6. We say a sequence of random variables, X_n, is **tight** or **bounded in probability**, if for every $\epsilon > 0$, there exists a number M such that for all n, $\mathrm{P}(|X_n| > M) < \epsilon)$. Show that if X_n is tight and $Y_n \xrightarrow{\mathrm{P}} 0$, then $X_n Y_n \xrightarrow{\mathrm{P}} 0$.