The Symmetric Exclusion Process: Correlation Inequalities and Applications

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Pemantle's problem (2000): Let η_t be a symmetric exlusion process on Z^1 with transition probabilities

$$p(x, y) = p(y, x) = p(y - x)$$

and $\eta_0 : \cdots\ 1\ 1\ 1\ 0\ 0\ 0\ \cdots$, and let

$$N_t = \sum_{x>0} \eta_t(x).$$

Is it true that

$$\frac{N_t - EN_t}{[var(N_t)]^{1/2}} \Rightarrow N(0, 1)?$$

Negative Correlations

Andjel (1988): If $A \cap B = \emptyset$, then

$$P^\eta(\eta_t \equiv 1 \text{ on } A, \eta_t \equiv 1 \text{ on } B) \leq P^\eta(\eta_t \equiv 1 \text{ on } A)P^\eta(\eta_t \equiv 1 \text{ on } B).$$

The same proof **does not** give

$$P^\eta(\eta_t \equiv 1 \text{ on } A, \eta_t \equiv 0 \text{ on } B) \geq P^\eta(\eta_t \equiv 1 \text{ on } A)P^\eta(\eta_t \equiv 0 \text{ on } B).$$

Now we know:

Theorem. *For any symmetric exclusion process with deterministic (or product) initial distribution,*

(a) η_t *is negatively associated (NA), i.e.,*

$$Ef(\eta_t)g(\eta_t) \leq Ef(\eta_t)Eg(\eta_t)$$

for all $f, g \uparrow$ *depending on disjoint sets of coordinates, and*

(b) $\forall T$, $\exists$ *independent Bernoulli random variables* $\zeta(x)$ *so that*

$$\sum_{x \in T} \eta_t(x) \qquad \text{and} \qquad \sum_{x \in T} \zeta(x)$$

have the same distribution.

Corollary. *If* $var(N_t) \to \infty$, *then* N_t *satisfies the CLT.*

The Strong Rayleigh Property

The generating polynomial of a p.m. μ on $\{0,1\}^n$ is

$$f(z_1, ..., z_n) = E_\mu \prod_{i=1}^{n} z_i^{\eta(i)}.$$

Then

$$\left.\frac{\partial f}{\partial z_i}\right|_{z_k \equiv 1} = E_\mu \eta(i), \quad \left.\frac{\partial^2 f}{\partial z_i \partial z_j}\right|_{z_k \equiv 1} = E_\mu \eta(i)\eta(j).$$

Pairwise negative correlations is equivalent to

$$(*) \qquad f(z)\frac{\partial^2 f}{\partial z_i \partial z_j}(z) \le \frac{\partial f}{\partial z_i}(z)\frac{\partial f}{\partial z_j}(z)$$

for $z_k \equiv 1$.

Definitions. (a) μ is **Strong Rayleigh** (SR) if (*) holds for all $z \in R^n$.

(b) μ is **stable** if $f \neq 0$ if $\Im(z_k) > 0$ for all k.

Remark. If $\mu = \nu_\alpha$ is a product measure, then

$$f(z) = \prod_{i=1}^{n} [\alpha_i z_i + (1 - \alpha_i)],$$

so μ is SR — (*) holds with equality — and stable.

Results about the Strong Rayleigh Property

Theorem. *(Brändén (2007)) SR is equivalent to stability.*

Why is this true? Think of it as an analogue of the quadratic formula for the roots of $ax^2 + bx + c$: Stability is a statement about whether there are roots in the upper half plane. SR is like a discriminant condition.

Theorem. *SR* $\implies$ *NA.*

Proof. Based on the Feder-Mihail (1992) proof of NA for the uniform spanning tree measure. Easier if $\sum_x \eta(x)$ is constant. Key use of SR property: If μ is SR on $\{0,1\}^n$, then so is its "symmetric homogenization" on $\{0,1\}^{2n}$, which satisfies $\sum_x \eta(x)$ constant.

Theorem. *If the initial distribution of a symmetric exclusion process is SR, then so is the distribution at time t.*

Proof. It is sufficient to prove it for exclusion on two sites, i.e. that stability is preserved by the transformation:

$$\mu \to T\mu = p\mu + (1-p)\mu_{i,j}.$$

($\mu_{i,j}$ is obtained from μ by permuting $\eta(i), \eta(j)$.)

Suppose f is stable. Need to show that if $\Im(z_k) > 0$ for all k, then $Tf(z) \neq 0$. Fix z_k for $k \neq i, j$. Need to show that T preserves stability of polynomials of the form $h(z,w) = a + bz + cw + dzw$. with complex a, b, c, d. If not all coefficients are zero, h is stable iff

$$\Re(b\overline{c} - a\overline{d}) \geq |bc - ad|,$$

$$\Im(a\overline{b}) \geq 0, \ \Im(a\overline{c}) \geq 0, \ \Im(b\overline{d}) \geq 0, \ \Im(c\overline{d}) \geq 0.$$

Theorem. *If the distribution of* $\{\eta(i), 1 \leq i \leq n\}$ *is SR, then there exist independent Bernoulli* $\{\zeta(i), 1 \leq i \leq n\}$ *so that*

$$\sum_i \eta(i) \qquad \textit{and} \qquad \sum_i \zeta(i)$$

have the same distribution.

Proof. $f(z, z, ..., z) = Ez^{\sum \eta(i)}$ is not zero if $Im(z) > 0$ or if $Im(z) < 0$ or if $z > 0$. So all roots are negative:

$$Ez^{\sum \eta(i)} = \prod_i [\alpha_i z + (1 - \alpha_i)],$$

where the roots are $-(1 - \alpha_i)/\alpha_i$.

Back to the CLT for the Exclusion Process

Theorem. *Suppose* $\sigma^2 = \sum_n n^2 p(n) < \infty$. *Then*

$$\frac{N_t - EN_t}{[var(N_t)]^{1/2}} \Rightarrow N(0,1).$$

Furthermore,

$$\lim_{t\to\infty} \frac{EN_t}{\sqrt{t}} = \frac{\sigma}{\sqrt{2\pi}} \quad \textit{and} \quad 0 < c_1 \le \frac{var(N_t)}{\sqrt{t}} \le c_2 < \infty.$$

Proof. Need to consider the first two moments of N_t. Let X_t, Y_t be independent copies of the random walk. By duality,

$$EN_t = \sum_{x>0} P(\eta_t(x) = 1) = \sum_{x>0} P^x(X_t \le 0) = E^0 X_t^+.$$

Similarly,

$$\sum_{x>0} \big[P(\eta_t(x) = 1)\big]^2 = E^{(0,0)} \min(X_t^+, Y_t^+).$$

So,

$$\sum_{x>0} var(\eta_t(x)) \sim \frac{\sigma}{2\sqrt{\pi}} \sqrt{t}.$$

Let

$$K(t) = -\sum_{x,y>0;x\neq y} cov(\eta_t(x)\eta_t(y)).$$

Then if $f(x,y) = 1_{\{x,y\leq 0\}}$,

$$\begin{aligned} K(t) &= \sum_{x,y>0;x\neq y} [U(t) - V(t)]f(x,y) \\ &= \sum_{x,y>0;x\neq y} \int_0^t V(t-s)(U-V)U(s)f(x,y)ds \\ &\leq \int_0^t \sum_{x<y} p(x,y)\big[P^0(x \leq X_s < y)\big]^2 \gamma(t-s,x,y)ds, \end{aligned}$$

where

$$\gamma(t,x,y) = P^0(X_t < x)P^0(X_t < y) + P^0(X_t \geq x)P^0(X_t \geq y).$$

Using $\gamma(t,x,y) \leq 1$ leads to

$$\limsup_{t\to\infty} \frac{K(t)}{\sqrt{t}} \leq \frac{\sigma}{2\sqrt{\pi}}.$$

Being more careful, one gets

$$\limsup_{t\to\infty} \frac{K(t)}{\sqrt{t}} < \frac{\sigma}{2\sqrt{\pi}}.$$

Poisson Convergence

Theorem. *Suppose the Bernoulli random variables* $\{\eta_n(x)\}$ *are strong Rayleigh for each* n*. If*

$$\lim_{n\to\infty}\sum_x E\eta_n(x)=\lambda,\qquad \lim_{n\to\infty}\sum_x[E\eta_n(x)]^2=0,$$

and

$$\lim_{n\to\infty}\sum_{x\neq y}Cov(\eta_n(x),\eta_n(y))=0,$$

then

$$\sum_x \eta_n(x)\Rightarrow Poisson(\lambda).$$

Application to Symmetric Exclusion

Recall that the extremal invariant measures μ_α are in one to one correspondence with harmonic functions $\alpha(x)$ for P with $0\le\alpha(x)\le 1$, and

$$\mu_\alpha=\lim_{t\to\infty}\nu_\alpha S(t).$$

Furthermore, $\mu_\alpha=\nu_\alpha$ iff α is constant.

Theorem. μ_α *is SR, and hence NA*

Example

Let P be simple random walk on the binary tree:

$l(x):$ 2 1 0 0 1 2

Theorem. *Suppose*

$$\alpha(x) = \begin{cases} \frac{1}{3\cdot 2^{l(x)}} & \text{if } x \in L, \\ 1 - \frac{1}{3\cdot 2^{l(x)}} & \text{if } x \in R. \end{cases}$$

Then with respect to μ_α,

$$\sum_{x\in L: l(x)=n} \eta(k) \Rightarrow \ \textit{Poisson } (1/3)$$

$$\sigma_n^{-1}\left[\sum_{x\in L: l(k)<n} \eta(x) - \frac{n}{3}\right] \Rightarrow N(0,1),$$

where $\frac{23}{189} \le \sigma_n^2/n \le \frac{1}{3}$ *asymptotically.*