Finitely Dependent Coloring on $\mathbb{Z}$ and other Graphs

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Schramm's Question: For which values of k, q does there exist a stationary $k-$dependent $q-$coloring of $\mathbb{Z}$?

This means: $X_i \in [q] = \{1, \ldots, q\}$ such that
(a) $(X_i) =^d (X_{i+1})$,
(b) $X_i \neq X_{i+1}$,
(c) $(\ldots, X_{i-2}, X_{i-1})$ and $(X_{i+k}, X_{i+k+1}, \ldots)$ are independent.

In 2008, Schramm knew:

(a) Impossible for $q = 2$, any k, since a stationary coloring must be

$$\begin{cases} \cdots 1212 \cdots & \text{with probability } \frac{1}{2}; \\ \cdots 2121 \cdots & \text{with probability } \frac{1}{2}. \end{cases}$$

(b) Impossible for $q = 3, k = 1$.
(c) Cannot be a block factor, i.e., $X_i = f(U_i, U_{i+1}, \ldots, U_{i+r-1})$, where U_i are i.i.d.
(d) Cannot be a Markov chain, or a function of a finite state Markov chain.

Based on these (and other) negative results, Schramm conjectured that the answer is always No.

However,

Theorem. There exists a stationary $1-$dependent $4-$coloring of $\mathbb{Z}$ and a stationary $2-$dependent $3-$coloring of $\mathbb{Z}$.

First, some of the negative results:

Proposition. There is no $k-$dependent $q-$coloring that is a Markov chain.

Proof. Suppose P is the transition matrix for the $[q]-$valued Markov chain X_i. By $k-$dependence, X_i is independent of X_0 for $i > k$. So, $(X_{k+1} \mid X_0) =^d (X_{k+2} \mid X_0) =$ the stationary distribution. Therefore, $P^{k+1}(I - P) = 0$, and the eigenvalues of P are 0 and 1. However, since X_i is a coloring, the diagonal elements of P are 0, so the trace is 0.

Proposition. There is no $1-$dependent $3-$coloring.

Proof. Fuxi Zhang observed that if X_i is a $1-$dependent $q-$coloring, then $1_{\{X_i=1\}}$ is a renewal sequence, e.g.,

$$\frac{P(000100)}{P(1)} = \frac{P(00-1-0)}{P(1)} = P(00)P(0),$$
$$\frac{P(0001)}{P(1)}\frac{P(100)}{P(1)} = \frac{P(00-1)}{P(1)}\frac{P(1-0)}{P(1)} = P(00)P(0).$$

The renewal time T has probability generating function

$$Es^T = \frac{ps^2}{1-s+ps^2},$$

where $p = P(X_0 = 1)$. Singularities are at $s = (1 \pm \sqrt{1-4p})/2p$. By Pringsheim's Theorem, $p \leq \frac{1}{4}$.

First Construction ($q = 4$).

Identify $\{1, 2, 3, 4\}$ with $\{-, +\}^2$, and write $X = \begin{pmatrix} Y \\ Z \end{pmatrix}$, where Y, Z are binary $\pm$ sequences. The distribution μ of Y is $1-$dependent. There are many possible choices for μ, e.g., Bernoulli($\frac{1}{2}$) and $Y_i = \text{sign}(U_i - U_{i-1}), \quad U_i$ i.i.d. $U[0, 1]$.

More generally, $1-$dependent binary sequences are determined by the sequence $u_n = \mu(+ + \cdots + +)$, where $n = \# +$'s, since e.g.,

$$\mu(+ - +) = u_1^2 - u_3.$$

The sequence u_n must satisfy many inequalities. A large collection can be described in terms of Polya frequency sequences.

Examples of the distribution of X:

$$2P\begin{pmatrix} + \\ z \end{pmatrix} = \mu(+), \quad 2^2 P\begin{pmatrix} + & - \\ z_1 & z_2 \end{pmatrix} = \mu(+-),$$

$$2^3 P\begin{pmatrix} + & - & + \\ z_1 & z_2 & z_3 \end{pmatrix} = \mu(+-+) - (-1)^{z_1+z_3}\mu(+++),$$

$$2^4 P\begin{pmatrix} + & - & + & - \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix} = \mu(+-+-)$$

$$-(-1)^{z_1+z_3}\mu(+++-) - (-1)^{z_2+z_4}\mu(+---),$$

$$2^5 P\begin{pmatrix} + & - & + & - & + \\ z_1 & z_2 & z_3 & z_4 & z_5 \end{pmatrix} = \mu(+-+-+)$$

$$-(-1)^{z_1+z_3}\mu(+++-+) - (-1)^{z_2+z_4}\mu(+---+)$$

$$-(-1)^{z_3+z_5}\mu(+-+++)$$

$$+(-1)^{z_1+z_5}[1+(-1)^{z_2+z_4}]\mu(+++++).$$

The choice $Y_i = \text{sign}(U_i - U_{i-1})$, U_i i.i.d. $U[0,1]$ is particularly nice, since for example

$$\begin{aligned}\mu(++-+-) &= P(U_0 < U_1 < U_2 > U_3 < U_4 > U_5) \\ &= \frac{\#\text{linear extensions of POS } 0 < 1 < 2 > 3 < 4 > 5}{\#\text{linear orders of } \{0,1,\ldots,5\}} \\ &= \frac{\alpha(2,1,1,1)}{6!}.\end{aligned}$$

Edelman, Hibi and Stanley (1989) proved (a more general version of) the recursion

$$\alpha(k_1, k_2, \ldots) = \alpha(k_1 - 1, k_2, \ldots) + \alpha(k_1, k_2 - 1, \ldots) + \cdots .$$

For example, for the partial orders

$$\begin{gathered} 0 < 1 < 2 > 3, \quad 0 < 1 > 2, \quad 0 < 1 < 2, \\ \alpha(2,1) = 3, \quad \alpha(1,1) = 2, \quad \alpha(2) = 1. \end{gathered}$$

The Formula.

$$P(x) = P\binom{y}{z} = \frac{1}{2^m(n+1)!}\sum_{w\in DD(m-1)}(-1)^{|w|}c(w,y,z)\alpha(y_w),$$

where

(a) x has length n and y has m runs.

(b) $DD(m)$ is the set of dispersed Dyck words of length m.

Examples:

$$+-+- \text{ and } ++-- \text{ are Dyck words;}$$

$$0++--00+-++-- \text{ is a dispersed Dyck word.}$$

(c) $|w|$ is the number of $+$'s in w and $c(w,y,z) = \pm 1$.

(d) y_w is obtained from y by eliminating runs in y according to w.

(e) $\alpha(y) = \#\pi \in S_{n+1}$ such that sign $(\pi_{i+1} - \pi_i) = y_i$.

Question: Why is $P(x) \geq 0$?

Second Construction ($q = 4$).

Write $P(x) = P(X_1 = x_1, \ldots, X_n = x_n)$ for the above coloring. Then the the following recursion holds, with $P(\emptyset) = 1$:

$$P(x) = \frac{1}{2(n+1)} \sum_{i=1}^{n} P(\widehat{x}_i), \quad x \in [4]^n, x_i \neq x_{i+1} \; \forall \; 1 \leq i < n,$$

where $\widehat{x}_i$ is obtained from x by deleting x_i. If $\widehat{x}_i$ is not a proper coloring, set $P(\widehat{x}_i) = 0$.

Consequence: $P(x) \geq 0$ for all x.

What happens if the analogous construction is applied to other q's?

(a) If $q = 3$, the coloring is 2−dependent.

(b) If $q \geq 5$, the coloring is not k−dependent for any k!

Some Details.

For any q and proper x, define

$$B(x) = \sum_{i=1}^{n} B(\widehat{x}_i).$$

Then for proper $x \in [q]^m, y \in [q]^n$,

$$\sum_{a\in[q]} B(xay) = 2\binom{m+n+2}{m+1} B(x)B(y), \qquad q = 4,$$

$$\sum_{a,b\in[q]} B(xaby) = 2\binom{m+n+4}{m+2} B(x)B(y), \qquad q = 3,$$

$$\sum_{x\in[q]^n} [B(1x2) - B(1x1)] = 2\prod_{k=1}^{n} [k(q-2) - 2], \qquad q \geq 2.$$

Color Symmetric Construction ($q \geq 4$).

Motivated by the second construction for $q = 4$, try

$$P(x) = \frac{1}{D(n+1)} \sum_{i=1}^{n} C(n-2i+1) P(\widehat{x}_i)$$

for $x \in [q]^n, x_i \neq x_{i+1}\ \forall\ 1 \leq i < n$. Motivated by special cases, take

$$C(n) = T_n(\sqrt{q}/2), n \geq 0; \quad D(n) = \sqrt{q}\, U_{n-1}(\sqrt{q}/2), n \geq 1,$$

where T_n, U_n are the Chebyshev polynomials of the first and second kind:

$$T_n(u) = \cosh(nt), \quad U_n(u) = \frac{\sinh[(n+1)t]}{\sinh(t)}, \quad u = \cosh(t).$$

C, D extended to all n by taking C even and D odd.
Note: If $q = 4$, $C(n) \equiv 1, \quad D(n) = 2n$.

Examples:

$$C(0)=1,\quad C(1)=\frac{\sqrt{q}}{2},\quad C(2)=\frac{q-2}{2},\quad C(3)=\frac{\sqrt{q}(q-3)}{2},$$

$$D(0)=0,\quad D(1)=\sqrt{q},\quad D(2)=q,\quad D(3)=\sqrt{q}(q-1).$$

Proof of 1−dependence relies on identities such as

$$2C(m)D(n)=D(m+n)+D(n-m)$$

and

$$C(j+k)D(k+l)=C(k)D(j+k+l)-C(l)D(j).$$

Also,

$$\sum_{i=1}^{n} D(n-2i+1)P(\widehat{x}_i)=0.$$

Bounds on the number of colors needed.

(a) On $\mathbb{Z}^1$, need 4 and this is best possible.

(b) On $\mathbb{Z}^2$, need at least 9 and 16 suffices.

(c) On $\mathbb{Z}^3$, need at least 12 and 64 suffices

(d) On the $d-$regular tree need at least de.

Construction of a $16-$coloring on $\mathbb{Z}^2$.

On each horizontal and vertical line in $\mathbb{Z}^2$, put an independent copy of the $\mathbb{Z}^1$ $4-$coloring. The color of $(m, n) \in \mathbb{Z}^2$ is (a, b), where a, b are the colors it inherits from the horizontal and vertical lines through it.

Five colors do not suffice on $\mathbb{Z}^2$.

Let $p = P(X_0 = 1)$ and

$$f(n) = P\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad g(n) = P\begin{pmatrix} 0 & 0 & \cdots & 0 & - \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where $n =$ the length of the strip and 0 means color other than 1. Then

$$f(n) = g(n) - pg(n-1), \quad g(n) = f(n-1) - pg(n-1).$$

Solving gives

$$\sum_{n=0}^{\infty} g(n)s^n = \frac{1}{1-(1-p)s+ps^2}.$$

By Pringsheim's Theorem, $p \leq 3 - 2\sqrt{2} = .171\ldots.$

Other results.

Combining our $4-$coloring with results in Holroyd, Schramm and Wilson, one can prove the existence of stationary $k-$dependent $q-$colorings that exclude certain other collections of patterns. (Colorings are those that exclude the pattern aa for any a.)

Some open problems.

1. Uniqueness of the $4-$coloring or symmetric $q-$coloring ($q \geq 5$) on $\mathbb{Z}$.

2. Existence of an automorphism invariant $q-$coloring on $\mathbb{Z}^2$.

3. Existence of an automorphism invariant $q-$coloring on the $d-$regular tree.

4. Do these colorings have any reasonable structure?