Some Open Problems

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For many years, David Aldous has maintained a list of open problems; see
http://www.stat.berkeley.edu/ aldous/Research/OP/index.html
This has led to a substantial amount of productive work in probability theory. I have decided to try to follow his excellent example. I trust that I will be informed of any progress made on these problems.

In [4], I posed approximately 65 problems, and then reported on the status of these problems in the 2005 reprinting of [4]. Quite a few are still open, and still of interest. Among these are #4 in Chapter I, # 2,7 in Chapter III, #1 in Chapter IV, #10,11,12,14 in Chapter VII, #5 in Chapter VIII.

1 The Exclusion Process

The exclusion process is an interacting particle system η_t taking values in $\{0,1\}^S$, where S is a countable set. There is at most one particle per site, and $\eta(x) = 0$ or 1 according to whether x is vacant or occupied. A particle at $x \in S$ attempts a transition to $y \in S$ at an exponential time with rate $p(x, y)$, where $p(x, y)$ are the transition probabilities for an irrreducible Markov chain on S. If y is vacant, it moves to y, while if y is occupied, it remains at x. For background material, see Chapter VIII of [4] and Part III of [5].

1. Stationary distributions in one dimension. Take $S = Z^1$ and $p(x, y) = p(y-x)$ satisfying

$$\sum_x |x|p(x) < \infty, \quad \sum_x xp(x) > 0, \quad \sum_{x<0} x^2 p(x) = \infty.$$

Prove or disprove that there exists a stationary distribution μ for the process satisfying

$$\lim_{x\to-\infty} \mu\{\eta : \eta(x) = 1\} = 0, \quad \lim_{x\to+\infty} \mu\{\eta : \eta(x) = 1\} = 1.$$

I don't know what to expect. The answer would be of interest even if the problem were solved for one example of $p(\cdot)$. For background on this problem, see [3].

2. Stationary distributions in higher dimensions. Take $S = Z^2$, for example, and $p(x,y) = p(y-x)$ given by

$$p(x) = \begin{cases} p_1 & \text{if } x = (1,0); \\ q_1 & \text{if } x = (-1,0); \\ p_2 & \text{if } x = (0,1); \\ q_2 & \text{if } x = (0,-1), \end{cases}$$

where $p_1 > q_1$ and $p_2 > q_2$. Show that if the angle between $\mathbf{v}$ and the mean vector $(p_1 - q_1, p_2 - q_2)$ is $< \frac{\pi}{2}$, then there exists a stationary distribution μ for the process that is invariant under shifts in the direction orthogonal to $\mathbf{v}$ (say, if $\mathbf{v}$ has rational coordinates) and satisfies

$$\lim_{n \to -\infty} \mu\{\eta : \eta(n\mathbf{v}) = 1\} = 0, \qquad \lim_{n \to +\infty} \mu\{\eta : \eta(n\mathbf{v}) = 1\} = 1.$$

This is known to be true in three cases:

$$\mathbf{v} = (0,1), \quad \mathbf{v} = (1,0), \quad \mathbf{v} = (\log p_1/q_1, \log p_2/q_2).$$

In fact, these are exactly the cases in which μ can be taken to be a product measure. For background on this problem, see [2].

3. The mean zero case. Take $S = Z^d$ and $p(x,y) = p(y-x)$ with $\sum_x xp(x) = 0$. Show that all stationary measures are exchangeable. This is known if $d = 1$ (Theorem 3.14 on page 391 of [4]) and in the symmetric case $p(-x) = p(x)$ for any d (Theorem 1.44 on page 377 of [4]).

4. Negative association. A probability measure μ on $\{0,1\}^S$ is said to be negatively associated (NA) if

$$\int fg d\mu \leq \int f d\mu \int g d\mu$$

for all increasing functions f, g on $\{0,1\}^S$ that depend on disjoint sets of coordinates. It was proved in [1] that if $p(x,y) = p(y,x)$ for all x, y and the initial distribution is deterministic, then the distribution of the process at all times is NA. This statement is trivially false for almost all asymmetric exclusion processes. However, at a workshop at IHP in Paris in 2008, I was asked whether it is true in the following special case: $S = Z^1$, $p(1) = p, p(-1) = q$ with $p > q$, and initially $\eta = \cdots 11110000 \cdots$. I don't know the answer, but it is plausible. (It is not true for the initial configuration $\eta = \cdots 00001111 \cdots$.)

References

[1] Borcea, J., Branden, P., and Liggett, T. M. (2009). *Negative dependence and the geometry of polynomials.* J. American Mathematical Society **22**, 521-567.

[2] Bramson, M. and Liggett, T. M. (2005). *Exclusion processes in higher dimensions: stationary measures and convergence.* Ann. Probab. **33**, 2255-2313.

[3] Bramson, M., Liggett, T. M., and Mountford, T. (2002).*Characterization of stationary measures for one dimensional exclusion processes.* Ann. Probab. **30**, 1529-1575.

[4] Liggett, T. M. (1985). *Interacting Particle Systems.* Springer

[5] Liggett, T. M. (1999). *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes.* Springer