ASYMPTOTICS OF A MATRIX VALUED MARKOV CHAIN ARISING IN SOCIOLOGY

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Abstract. We consider a discrete time Markov chain whose state space is the set of all $N \times N$ stochastic matrices with zero diagonal entries. This chain models the evolution of relationships among N individuals who exchange gifts according to probabilities determined by previous exchanges. We determine the stable equilibria for this chain, and prove convergence to a mixture of these. In particular, we show that for generic initial states, the chain converges to a randomly chosen set of constellations made up of disjoint stars. Each star has a center, which is the recipient of all gifts from the other individuals in that star, while the center distributes his gifts only to members of his own star.

1. Introduction. The study of how power differences emerge when actors negotiate for advantage within exchange networks has been a thriving and lively area within social psychology, and models have been developed which predict quite well the patterns of exchange and negotiating power (Cook *et. al.* (1983), Markovsky *et. al.* (1993), Bonacich (1998)). However, although there is a body of experimental work on power and exchange patterns in non-negotiated exchange networks, networks governed by reciprocity rather than binding agreements (Molm *et. al.* (2000)), there is no general model for predicting exchange patterns and power differences between positions for these networks.

To put our model in some context, we note that in his book, Peter Blau (1967) distinguished between economic and social exchange, as exemplified by a favor or a gift. The partners to an economic exchange bargain over its terms and trust is not necessary because the exchange is visible and simultaneous. A favor or a gift, however, may never be reciprocated and there is no overt bargaining. This

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distinction mirrors the game theorist's distinction between cooperative and non-cooperative games; in cooperative games actors can come to binding agreements about their behavior and the distribution of rewards from their interaction, whereas in non-cooperative games they cannot. The model in this paper is designed to mirror situations in the real world in which individuals occasionally do favors for one another. Unlike game theory, we assume that individuals are motivated by a norm of reciprocity, not self-interest.

We will consider here a model for situations in which individuals reciprocate gifts from others in proportion to the frequency with which they have been rewarded in the past and the value of the reward. A randomly chosen actor i selects another actor j with probability $p_{i,j}$ for a reward $c_{i,j}$. This reward increases the probability $p_{j,i}$ that j will choose i in a later round. In this paper, we determine the asymptotic behavior of the model as time tends to infinity.

This project began with extensive simulations of various networks by the first author, in an attempt to understand the limiting behavior of the system. These simulations suggested that certain equilibria are unstable. Our rigorous analysis shows that these are in fact stable, although the probability of convergence to them is presumably so small that it is not observed in simulations unless the initial position is chosen very carefully.

The model we consider here is a discrete time Markov chain whose state space is the set of all $N \times N$ stochastic matrices $P = (p_{i,j})$ with $p_{i,i} = 0$ for each i. The transition probabilities for this Markov chain are given in terms of a collection of numbers $c_{i,j}, 1 \leq i \neq j \leq N$ that satisfy:

(a) $0 \leq c_{i,j} < 1$ for each i, j,

and

(b) $c_{i,j} > 0$ if and only if $c_{j,i} > 0$, (quasi-symmetry)

The interpretation is that $c_{i,j}$ is the size of the gift that i gives j when i chooses j as his recipient. Given the state $P(n)$ of the chain at time n, $P(n+1)$ is obtained as follows. Choose a row i of $P(n)$ with probability $\frac{1}{N}$, and then a column j with probability $p_{i,j}(n)$. Then

$$p_{k,l}(n+1) = \begin{cases} p_{k,l}(n) & \text{if } k \neq j, \\ p_{j,i}(n) + c_{i,j}[1 - p_{j,i}(n)] & \text{if } k = j, l = i \\ (1 - c_{i,j}) p_{j,l}(n) & \text{if } k = j, l \neq i. \end{cases} \tag{1.1}$$

Perhaps the most natural case of our model is the symmetric one, in which $c_{i,j} = c_{j,i}$ for all i, j. However, we assume only quasi-symmetry, since this generalization does not require more difficult arguments. The presentation would be simplified if we assumed $c_{i,j} > 0$ for all i, j, but that assumption would eliminate many examples with interesting structure – see the final example in this section.

The rule in (1.1) above is reminiscent of a model developed by Bush and Mosteller (1955) to describe stochastic learning, although our situation is quite different: in learning actions occur before rewards, but in reciprocity rewards from others occur before reciprocated actions. Generalizations of the Bush-Mosteller model have been studied in both the mathematics and applied literature – see, for example, Iosifescu and Grigorescu (1990) and the references there. Most general theorems in the field appear to be devoted to uniquely ergodic systems, and thus do not apply to our situation. In other related work, Burton and Keller (1993) give an analysis of a model on the simplex $\{x = (x_1, ..., x_N) : x_i \geq 0 \; \forall \; i, \sum_{i=1}^N x_i = 1\}$ with an evolution analogous to ours. They show that under appropriate conditions, the extremal stationary distributions concentrate on vertices or edges of the simplex. In fact our model falls within the general framework of "randomly chosen maps" that is considered in that paper.

After submission of the original version of this paper, the authors became aware of a recent paper by Pemantle and Skyrms (2000), which treats a similar class of models. The *a priori* structure given by our $c_{i,j}$'s is absent in their model, and the models differ in other ways as well. For example, in (1.1), the choice of (i, j) at a given time reinforces choices of (j, i) at later times. In the Pemantle and Skyrms models the reinforcement is either to (i, j) choices ("Friends I") or symmetrically to (i, j) and (j, i) ("Friends II"). Nevertheless, our results for our model are very similar to their results in the case of their Friends II model with discounting. (See their Theorem 4.)

Returning to our model, the limiting behavior of $P(n)$ as $n \to \infty$ will be described in terms of what we will call "stars" and "constellations". The matrix P will be called a star with center a if $p_{i,a} = 1$ for all $i \neq a$ (and therefore $p_{k,l} = 0$ for all $k, l \neq a$). More generally, we will say that P is a constellation with centers $a_1, ..., a_k$ and regions of influence $A_1, ..., A_k$ if $A_1, ..., A_k$ is a partition of $\{1, ..., N\}$ with $a_1 \in A_1, ..., a_k \in A_k$ so that for each l the matrix $(p_{i,j}; i, j \in A_l)$ is a (stochastic) star with center a_l. The "structure" of the constellation is the partition $A_1, ..., A_k$, together with the collection of centers $a_1, ..., a_k$. Note that if $P(0)$ is a constellation, then $P(n)$ is a constellation with the same structure for every $n \geq 1$. Furthermore, even if $P(n)$ consists of a single star with center a, the probabilities $p_{a,i}(n)$ evolve in an interesting way – see the proof of Proposition 2.1.

In Section 2, we will study the chain $P(n)$ restricted to constellations of a given structure, while in Section 3, we will show that these sets of constellations are stable, in the sense that the chain converges to them with positive probability from most initial states, provided that $c_{i,a_l} > 0$ for all $i \in A \backslash \{a_l\}$ and all l. In Section 4, we show that the chain must converge a.s. to a randomly chosen set of constellations (again, for most initial states). The results of these sections are summarized in the theorem below. For its statement, we need some terminology. If $\mathcal{C} = \mathcal{C}(A_1, ..., A_k; a_1, ..., a_k)$ is the set of constellations with structure $A_1, ..., A_k; a_1, ..., a_k$, we will say that $P(n) \to \mathcal{C}$ if for each $1 \leq l \leq k$,

$$\lim_{n\to\infty} p_{i,a_l}(n) = 1 \quad \text{for all } i \in A_l \backslash \{a_l\} \quad \text{and} \quad \lim_{n\to\infty} \sum_{j \in A_l} p_{a_l,j}(n) = 1.$$

Probabilities computed relative to the chain with initial state P will be denoted by $\mathcal{P}^P$. In order to simplify some expressions here and in later sections, we will define $c_{i,i} = 0$ for each i.

Theorem 1.2. *(a) Suppose that for each $1 \leq i \leq N$, $p_{i,j} > 0$ for some $1 \leq j \leq N$*

with $c_{i,j} > 0$. *Then*

$$\mathcal{P}^P\big(P(n) \to \mathcal{C}(A_1, ..., A_k; a_1, ..., a_k) \text{ for some structure } \{A_i, a_i\}_{i=1}^k\big) = 1.$$

(b) Suppose that for each $1 \leq l \leq k$, $c_{i,a_l} > 0$ *for some* $i \in A_l$. *Then a.s. on the event* $\{P(n) \to \mathcal{C}(A_1, ..., A_k; a_1, ..., a_k)\}$, $P(n)$ *converges in distribution to a limit* $P(\infty)$ *where*

$$Ep_{a_l,i}(\infty) = \frac{c_{i,a_l}}{\sum_{j\in A_l} c_{j,a_l}}, \quad i \in A_l.$$

(c) For each structure $A_1, ..., A_k; a_1, ..., a_k$,

$$\mathcal{P}^P\big(P(n) \to \mathcal{C}(A_1, ..., A_k; a_1, ..., a_k)\big) > 0 \tag{1.3}$$

if and only if for every $1 \leq l \leq k$, *(i)* $p_{i,a_l} = 1$ *for each* $i \in A_l\backslash\{a_l\}$ *with* $c_{i,a_l} = 0$, *(ii)* $\sum_{j\in A_l} p_{a_l,j} = 1$ *if* $c_{i,a_l} = 0$ *for all* $i \in A_l$, *and (iii)* $p_{i,a_l} + p_{a_l,i} > 0$ *for all* $i \in A_l\backslash\{a_l\}$.

Part (a) of the theorem is proved in Section 4. Part (b) is proved in Section 2. We give there also a set of equations that determines all of the moments of $p_{a_l,i}(\infty)$. The final part of the theorem is proved in Section 3. An explicit lower bound for the probability in (1.3) is provided there.

As an example, consider the case $N = 3$. Then there are unique stationary random stars of the three forms

$$\begin{pmatrix} 0 & X & 1-X \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ Y & 0 & 1-Y \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ Z & 1-Z & 0 \end{pmatrix}, \tag{1.4}$$

where

$$EX = \frac{c_{2,1}}{c_{2,1} + c_{3,1}}, \quad EY = \frac{c_{1,2}}{c_{1,2} + c_{3,2}}, \quad EZ = \frac{c_{1,3}}{c_{1,3} + c_{2,3}},$$

provided that the denominators are strictly positive. In fact, if $c_{2,1} = c_{3,1} = \frac{1}{2}$, one can say more: X is uniformly distributed on $[0, 1]$. If all the $c_{i,j}$'s are strictly positive, then for any initial state $P(0)$, the chain converges in distribution to a mixture of these three equilibria. (The convergence to the form of the limit is a.s.) Each of the three limiting forms has positive probability in the limit if $p_{i,j}(0) + p_{j,i}(0) > 0$ for all $i \neq j$. If $c_{2,3} = c_{3,2} = 0$ and the other $c_{i,j}$'s are strictly positive, then $Y = Z = 1$. If $p_{2,1}(0) > 0$ and $p_{3,1}(0) > 0$, then there is a.s. convergence to the first star in (1.4).

Another example is a ring of size N. Here $c_{i,j} > 0$ if i and j are neighbors on the ring ($|i - j| = 1$ or $|i - j| = N - 1$), and $c_{i,j} = 0$ otherwise. Then in the limit, one sees a randomly chosen constellation whose stars are connected intervals (of lengths 2 and 3) on the ring, provided that for each i, $p_{i,j}(0) > 0$ for some neighbor j of i. If $p_{i,j}(0) > 0$ for all pairs of neighbors i, j, then each such constellation has positive probability of appearing the limit.

2. The chain on sets of constellations. In this section, we fix a partition $A_1, ..., A_k$ of $\{1, 2, ..., N\}$ and centers $a_1 \in A_1, .., a_k \in A_k$. In the first part of the section, we consider the chain $P(n)$ on the set $\mathcal{C}$ of constellations with this structure. (Later, we consider what happens on the event $\{P(n) \to \mathcal{C}\}$.) In the first result, we will see that this chain is (usually) ergodic, and that the moments of the stationary distribution can be computed explicitly.

Proposition 2.1. *Suppose that* $\sum_{i \in A_l} c_{i,a_l} > 0$ *for each* l. *Then the Markov chain* $P(n)$ *restricted to* $\mathcal{C}$ *has a unique stationary distribution, and* $P(n)$ *converges weakly to it for any initial state in* $\mathcal{C}$. *For the limiting distribution* $P(\infty)$,

$$Ep_{a_l,i}(\infty) = \frac{c_{i,a_l}}{\sum_{i' \in A_l} c_{i',a_l}} \tag{2.2}$$

for $i \in A_l \backslash \{a_l\}$.

Proof. The proof is by the method of moments. A byproduct of the proof is a set of equations that uniquely determines the moments of the stationary distribution. The choice of (i, j) in the evolution of $P(n)$ on $\mathcal{C}$ can be thought of as first choosing l with probability $|A_l|/N$ (where $|\cdot|$ denotes cardinality), then choosing $i \in A_l$ with probability $1/|A_l|$, and finally choosing j with probability $p_{i,j}$. Therefore it suffices to consider constellations consisting of a single star.

So, suppose $P(n)$ is the Markov chain on stars with center 1. Then $P(n)$ is of the form

$$P(n) = \begin{pmatrix} 0 & X_2(n) & X_2(n) & \cdots & X_N(n) \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and $X(n) = (X_2(n), ..., X_N(n))$ is a Markov chain on a simplex with the following transitions: There is no change with probability $1/N$, and for each $2 \le i \le N$,

$$X_j(n+1) = \begin{cases} X_j(n) + c_{i,1}[1 - X_j(n)] & \text{if } j = i, \\ X_j(n)(1 - c_{i,1}) & \text{if } j \ne i \end{cases}$$

with probability $1/N$. Therefore, letting $\mathcal{F}_n$ be the σ-algebra generated by the process up to time n,

$$NE\left[\prod_{i=2}^{N} X_i^{m_i}(n+1) \middle| \mathcal{F}_n\right] = \prod_{i=2}^{N} X_i^{m_i}(n) + \sum_{j=2}^{N} \big(X_j(n) + c_{j,1}[1 - X_j(n)]\big)^{m_j} \times \prod_{i \ne j} (1 - c_{j,1})^{m_i} X_i^{m_i}(n)$$

for nonnegative integers $m_2, ..., m_N$. Using the binomial expansion for the term

$$\big(X_j(n) + c_{j,1}[1 - X_j(n)]\big)^{m_j} = \big(c_{j,1} + (1 - c_{j,1})X_j(n)\big)^{m_j}$$

and taking expected values of both sides, we see that

$$
\begin{aligned}
NE\prod_{i=2}^{N} X_i^{m_i}(n+1) = &E\prod_{i=2}^{N} X_i^{m_i}(n) + \sum_{j=2}^{N}(1-c_{j,1})^m \sum_{l=0}^{m_j}\binom{m_j}{l}\left(\frac{c_{j,1}}{1-c_{j,1}}\right)^{m_j-l} \\
&\times EX_j^l(n)\prod_{i\neq j} X_i^{m_i}(n),
\end{aligned}
$$

where $m=\sum_j m_j$ is the total order of the moment on the left. Note that the total order of each of the moments appearing on the right is at most m, and is equal to m exactly when $l=m_j$. Therefore

$$
NE\prod_{i=2}^{N} X_i^{m_i}(n+1) = \left[1+\sum_{j=2}^{N}(1-c_{j,1})^m\right] E\prod_{i=2}^{N} X_i^{m_i}(n) + \text{ lower order moments.}
$$

After dividing this equality by N, it is of the form

$$
f(n+1)=\gamma f(n)+g(n)
$$

with $0<\gamma<1$, and in arguing recursively on the total order of the moment, we will know that $\lim_n g(n)$ exists, and will want to conclude that $\lim_n f(n)$ exists. To see this, iterate the expression above that relates f to g to obtain

$$
f(n)=\gamma^n f(0)+\sum_{k=0}^{n-1}\gamma^k g(n-1-k),
$$

and hence $\lim_n f(n)=(1-\gamma)^{-1}\lim_n g(n)$ by the dominated convergence theorem.

We therefore can conclude that

$$
M(m_2,...,m_N)=\lim_{n\to\infty} E\prod_{i=2}^{N} X_i^{m_i}(n)
$$

exists, and satisfies

$$
\begin{aligned}
(N-1)M(m_2,...,m_N) = &\sum_{j=2}^{N}(1-c_{j,1})^m \sum_{l=0}^{m_j}\binom{m_j}{l}\left(\frac{c_{j,1}}{1-c_{j,1}}\right)^{m_j-l} \\
&\times M(m_2,...,l,...,m_N),
\end{aligned} \tag{2.3}
$$

where the l is in the jth position in the vector $(m_2,...,l,...,m_N)$ on the right. Expression (2.2) is obtained from these equations when $m=1$.

Remark. In some cases, one can determine the limiting distribution explicitly by checking that the moments of a candidate distribution satisfy (2.3). For example, if $N=3$ and $c_{2,1}=c_{3,1}=\frac{1}{2}$, the stationary random star with center 1 has $p_{1,2}$ uniformly distributed on $[0,1]$.

We now consider the chain $P(n)$ on the set of all matrices, and show that if $\mathcal{P}(P(n)\to\mathcal{C})>0$, then conditional on this event, the distribution of $P(n)$ converges to the stationary distribution identified in Proposition 2.1. This is intuitive, but does require some proof. To simplify the notation, we will consider the convergence of first moments only. The general case follows by induction, as in the proof of Proposition 2.1.

Proposition 2.4. *Suppose that $\sum_{i\in A_l} c_{i,a_l} > 0$ for each l. If $\mathcal{P}(P(n)\to\mathcal{C})>0$, then*

$$\lim_{n\to\infty} E\big(p_{a_l,i}(n)\mid P(m)\to\mathcal{C}\big)=\frac{c_{i,a_l}}{\sum_{i'\in A_l} c_{i',a_l}}$$

for $i\in A_l$.

Proof. For fixed $i\in A_l\backslash\{a_l\}$, the recursion takes the form

$$E\big[p_{a_l,i}(n+1)\mid \mathcal{F}_n\big]=\gamma p_{a_l,i}(n)+\sigma+\Delta(n), \tag{2.5}$$

where

$$\gamma=1-\frac{1}{N}\sum_{j\in A_l\backslash\{a_l\}} c_{j,a_l}<1,\quad \sigma=\frac{1}{N}c_{i,a_l},$$

and

$$\begin{aligned}\Delta(n)=&\frac{1}{N}p_{a_l,i}(n)\sum_{j\neq i} c_{j,a_l}[1_{\{j\in A_l\backslash\{a_l\}}-p_{j,a_l}(n)]\\ &-\frac{1}{N}c_{i,a_l}[1-p_{i,a_l}(n)][1-p_{a_l,i}(n)].\end{aligned}$$

Note that

$$\lim_{n\to\infty}\Delta(n)=0 \text{ a.s. on } \{P(m)\to\mathcal{C}\}. \tag{2.6}$$

Iterating (2.5) leads to

$$\begin{aligned}E\big[p_{a_l,i}(n)\mid\mathcal{F}_m\big]=&\sigma\sum_{j=0}^{n-m-1}\gamma^j+\gamma^{n-m}p_{a_l,i}(m)\\ &+\sum_{j=0}^{n-m-1}\gamma^j E[\Delta(n-j-1)\mid\mathcal{F}_m]\end{aligned} \tag{2.7}$$

for $m\le n$.

Let $A=\{P(n)\to\mathcal{C}\}$, and choose $A_m\in\mathcal{F}_m$ so that $\mathcal{P}(A\triangle A_m)\to 0$, where here $\triangle$ refers to the symmetric difference. Multiply (2.7) by the indicator of A_m and take expected values to get

$$\begin{aligned}E\big[p_{a_l,i}(n),A_m\big]=&\sigma\mathcal{P}(A_m)\sum_{j=0}^{n-m-1}\gamma^j+\gamma^{n-m}E[p_{a_l,i}(m),A_m]\\ &+\sum_{j=0}^{n-m-1}\gamma^j E[\Delta(n-j-1),A_m].\end{aligned}$$

Using (2.6) and the fact that $|\Delta(n)|\le 1$, we see that

$$\limsup_{n\to\infty}\left|E\big[p_{a_l,i}(n),A_m\big]-\frac{\sigma}{1-\gamma}\mathcal{P}(A_m)\right|\le\frac{1}{1-\gamma}\mathcal{P}(A_m\backslash A).$$

Letting $m \to \infty$ leads to

$$\lim_{n\to\infty} E\left[p_{a_l,i}(n), A\right] = \frac{\sigma}{1-\gamma}\mathcal{P}(A),$$

as required.

3. Stability of sets of constellations. In this section, we again fix a partition $A_1, ..., A_k$ of $\{1, 2, ..., N\}$ and centers $a_1 \in A_1, ..., a_k \in A_k$. We will show that for most initial states, the chain converges to the set of constellations with this structure with positive probability, provided that relevant $c_{i,j}$'s are strictly positive. Let μ be the distribution of the chain on the set of infinite paths $\{P(0), P(1), ...\}$, and let μ_n be the corresponding distribution on paths $\{P(0), P(1), ..., P(n)\}$ of length n.

We will prove convergence to constellations with the given structure with positive probability by proving that this convergence occurs for another chain with probability 1. The result is transferred back to the original chain via an analysis of the Radon-Nikodym derivative between the distributions of the two chains.

The comparison chain has the same potential transitions as $P(n)$, but they occur with different probabilities (many of which are zero). Specifically, the transition in (1.1) occurs with probability zero unless $i, j \in A_l$ for some l, and exactly one of i and j is a_l. In this case, the probability of the transition is

$$\frac{1}{N} \times \begin{cases} 1 & \text{if } i \in A_l \backslash \{a_l\}, j = a_l \\ p_{i,j}(n)/\sum_{j'\in A_l} p_{i,j'}(n) & \text{if } i = a_l, j \in A_l\backslash\{a_l\} \end{cases}$$

instead of $\frac{1}{N}p_{i,j}(n)$. We may assume that the denominator above is strictly positive, since otherwise the bound we will obtain is trivial.

Let ν and ν_n be the analogues of μ and μ_n for the new chain (with the same initial state). We wish to determine when $\nu_n << \mu_n$ and then write out an expression for the the Radon-Nikodym derivative of ν_n with respect to μ_n. A given sequence $\{P(0), ..., P(n)\}$ is the result of a sequence of choices

$$(i_0, j_0), ..., (i_{n-1}, j_{n-1}).$$

With this identification, $\nu_n << \mu_n$ and

$$\frac{d\nu_n}{d\mu_n}(P(0), ..., P(n)) = \prod_{l=1}^{k} \prod_{\substack{0\le m<n \\ i_m\in A_l\backslash\{a_l\} \\ j_m=a_l}} \left[p_{i_m,j_m}(m)\right]^{-1} \prod_{\substack{0\le m<n \\ i_m=a_l \\ j_m\in A_l\backslash\{a_l\}}} \left[\sum_{j'\in A_l} p_{i_m,j'}(m)\right]^{-1}$$

a.s. (ν_n), provided the right side above is finite. (There is a small abuse of notation here, since the sequence $P(n)$ may not determine the sequence (i_n, j_n), but considering this would only complicate the presentation.) Let

$$f(P(0), P(1), ...) = \prod_{l=1}^{k} \prod_{\substack{m\ge 0 \\ i\in A_l\backslash\{a_l\} \\ j=a_l}} \left[p_{i,j}(m)\right]^{-1} \prod_{\substack{m\ge 0 \\ i=a_l}} \left[\sum_{j'\in A_l} p_{i,j'}(m)\right]^{-1}. \tag{3.1}$$

Then

$$\frac{d\nu_n}{d\mu_n} \uparrow \qquad \text{and} \qquad \frac{d\nu_n}{d\mu_n} \leq f$$

a.s. (ν), so $\nu << \mu$ and

$$\frac{d\nu}{d\mu} \leq f \tag{3.2}$$

provided $f < \infty$ a.s. (ν). (See Theorem 3.3 of Chapter 4 of Durrett (1996), for example.)

We need to estimate the factors that appear in the expression for f in (3.1) for the comparison chain, in order to determine when $f < \infty$ a.s. (ν). With respect to the chain with distribution ν, if $i \in A_l \backslash \{a_l\}$, $p_{i,a_l}(m)$ has only one possible transition:

$$p_{i,a_l}(m+1) = p_{i,a_l}(m) + c_{a_l,i}\big[1 - p_{i,a_l}(m)\big] \tag{3.3}$$

with probability

$$\frac{1}{N} \frac{p_{a_l,i}(m)}{\sum_{j \in A_l} p_{a_l,j}(m)}.$$

On the other hand, the last factor on the right of (3.1) can change if any row in $A_l \backslash \{a_l\}$ is chosen. This happens with probability $(|A_l| - 1)/N$, and then the new value of the sum is given by

$$\sum_{j \in A_l} p_{a_l,j}(m+1) = \sum_{j \in A_l} p_{a_l,j}(m) + c_{i,a_l}\left[1 - \sum_{j \in A_l} p_{a_l,j}(m)\right], \tag{3.4}$$

where i is the row in $A_l \backslash \{a_l\}$ that was chosen. (The summands above can both increase and decrease, but the sum can only increase.)

The transitions in (3.3) and (3.4) are both of the form

$$x_{m+1} = x_m + c(1 - x_m).$$

The solution to this recursion is

$$x_m = 1 - (1-c)^m (1 - x_0).$$

We can bound (3.4) below by just considering transitions corresponding to the i which maximizes the value of c_{i,a_l} for $i \in A_l$. Therefore,

$$\sum_{j \in A_l} p_{a_l,j}(m) \geq_d 1 - \big(1 - \max_{i \in A_l} c_{i,a_l}\big)^{X_m} \left[1 - \sum_{j \in A_l} p_{a_l,j}(0)\right], \tag{3.5}$$

where X_m is binomially distributed with parameters m and $1/N$. (We use $\geq_d$ to mean that the distribution on the left is stochastically larger than the one on the right.)

Since the function $\log u/(1-u)$ is increasing on $(0,1]$,

$$\frac{\log u}{1-u} \geq \frac{\log u_0}{1-u_0}, \quad 0 < u_0 \leq u \leq 1. \tag{3.6}$$

The left side of (3.5) is increasing in m. So, letting u be the sum on the left side of (3.5) and u_0 be the corresponding sum when $m=0$, we can take logs in (3.5) and sum on m to get

$$\begin{aligned}\int \sum_{m=0}^{\infty} \log\Big[\sum_{j\in A_l} p_{a_l,j}(m)\Big] d\nu &\geq \log\Big[\sum_{j\in A_l} p_{a_l,j}(0)\Big] \sum_{m=0}^{\infty} E\big(1-c_l\big)^{X_m} \\ &= \log\Big[\sum_{j\in A_l} p_{a_l,j}(0)\Big] \frac{N}{c_l},\end{aligned} \tag{3.7}$$

where $c_l = \max_{i\in A_l} c_{i,a_l}$. Note that (3.7) is correct even if $c_l = 0$, provided we interpret $d/0$ as being $-\infty$ if $d<0$, and 0 if $d=0$.

For the comparison chain, consider now the conditional process, given

$$\{p_{a_l,i}(n), i \in A_l, n \geq 0\}.$$

Using the same estimates that led to (3.7), we see that for $i \in A_l \backslash \{a_l\}$,

$$\begin{aligned}&E\big[\log p_{i,a_l}(m) \mid p_{a_l,j}(n), j \in A_l, n \geq 0\big] \geq \\ &\qquad \log p_{i,a_l}(0) \prod_{n=0}^{m-1} \left[1 - \frac{c_{a_l,i} p_{a_l,i}(n)}{N \sum_{i'\in A_l} p_{a_l,i'}(n)}\right].\end{aligned} \tag{3.8}$$

At each time, there is probability $1/N$ of choosing the pair (i, a_l), and if this pair is chosen, the new value of $p_{a_l,i}$ is at least c_{i,a_l}, independently of the past. Therefore, the expected value of the product on the right of (3.8) is at most

$$\left[1 - \frac{c_{a_l,i} p_{a_l,i}(0)}{N \sum_{i'\in A_l} p_{a_l,i'}(0)}\right] \left[1 - \frac{c_{a_l,i} c_{i,a_l}}{N^2}\right]^{m-1}.$$

Taking expected values in (3.8) and summing on m then gives

$$\int \sum_{m=0}^{\infty} \log p_{i,a_l}(m) d\nu \geq \log p_{i,a_l}(0) \left[1 - \frac{c_{a_l,i} p_{a_l,i}(0)}{N \sum_{i'\in A_l} p_{a_l,i'}(0)}\right] \frac{2N^2}{c_{a_l,i} c_{i,a_l}}. \tag{3.9}$$

Again, this is correct even if $c_{a_l,i} = 0$ with our earlier convention.

Combining (3.1), (3.7) and (3.9), we have the following:

$$\int \log f d\nu \leq -N^2 \sum_{l=1}^{k} \left[2 \sum_{i\in A_l\backslash\{a_l\}} \frac{\log p_{i,a_l}(0)}{c_{a_l,i} c_{i,a_l}} + \frac{\log \sum_{j\in A_l} p_{a_l,j}(0)}{\max_{i\in A_l} c_{i,a_l}}\right]. \tag{3.10}$$

It follows that if the right side of (3.10) is finite, then $f < \infty$ a.s. (ν), so $\nu << \mu$. This leads to an explicit lower bound for the probability that the original chain converges to the set $\mathcal{C}$ of constellations with structure $A_1, ..., A_k; a_1, ..., a_k$, as we now see.

Proposition 3.11. *Suppose that for each* $1 \leq l \leq k$, $p_{i,a_l}(0) = 1$ *for each* $i \in A_l \backslash \{a_l\}$ *for which* $c_{i,a_l} = 0$, *and* $\sum_{j \in A_l} p_{a_l,j}(0) = 1$ *if* $c_{i,a_l} = 0$ *for all* $i \in A_l$. *Then*

$$-\log \mu\big(\{P(n) \to \mathcal{C}\}\big) \leq -N^2 \sum_{l=1}^{k} \left[2 \sum_{i \in A_l \backslash \{a_l\}} \frac{\log p_{i,a_l}(0)}{c_{a_l,i} c_{i,a_l}} + \frac{\log \sum_{j \in A_l} p_{a_l,j}(0)}{\max_{i \in A_l} c_{i,a_l}}\right].$$

If in addition $p_{i,a_l}(0) + p_{a_l,i}(0) > 0$ *for all* $i \in A_l \backslash \{a_l\}$, *then* $\mu\big(\{P(n) \to \mathcal{C}\}\big) > 0$.

Proof. We may assume in proving the final statement that the right side of (3.10) is finite. To see this, first recall that in deriving (3.10), we have made the convention that $0/0 = 0$. Secondly, note that if $p_{i,a_l}(0) = 0$ for some $i \in A_l \backslash \{a_l\}$, which would make the right side of (3.10) infinite, then we would have by assumption that $p_{a_l,i}(0) > 0$, and hence that $p_{i,a_l}(1) > 0$ with positive probability (since then, by assumption, $c_{i,a_l} > 0$). A similar remark applies to the case in which $\sum_{j \in A_l} p_{a_l,j}(0) = 0$.

Since $f < \infty$ a.s. (ν) by (3.10), the terms in the product in (3.1) tend to 1 as $m \to \infty$ a.s. (ν). Therefore $\nu\big(\{P(n) \to \mathcal{C}\}\big) = 1$. Using again the fact that $f < \infty$ a.s. (ν), $\nu << \mu$, so we may let $g = \frac{d\nu}{d\mu}$ and write

$$1 = \nu\big(\{P(n) \to \mathcal{C}\}\big) = \int_{\{P(n) \to \mathcal{C}\}} g d\mu \leq \left(\int g^p d\mu\right)^{\frac{1}{p}} \big(\mu(\{P(n) \to \mathcal{C}\})\big)^{\frac{1}{q}},$$

for $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$,where the inequality is an application of Hölder's inequality. Therefore,

$$\big[\mu(\{P(n) \to \mathcal{C}\})\big]^{p-1} \geq \left(\int g^p d\mu\right)^{-1}.$$

Subtracting $1 = \int g d\mu$ from both sides, dividing by $p - 1$, and letting $p \downarrow 1$ leads to

$$-\log \mu\big(\{P(n) \to \mathcal{C}\}\big) \leq \int g \log g d\mu = \int \log g d\nu \leq \int \log f d\nu$$

by (3.2). Now apply (3.10).

Remarks. (i) The sufficient conditions for $\mu\big(\{P(n) \to \mathcal{C}\}\big) > 0$ in Proposition 3.11 are easily seen to be necessary as well. Suppose, for example, that $p_{i,a_l}(0) < 1$ for some l and some $i \in A_l \backslash \{a_l\}$ with $c_{i,a_l} = 0$. Since $c_{i,a_l} = 0$, $p_{i,a_l}(n)$ is nonincreasing in n, and hence cannot converge to 1. Similarly, if $p_{i,a_l}(0) = p_{a_l,i}(0) = 0$ for some $i \in A_l \backslash \{a_l\}$, then $p_{i,a_l}(n) = 0$ for all n, and hence cannot converge to 1.

(ii) The fact that an explicit estimate for the probability of $\{P(n) \to \mathcal{C}\}$ is available will be important in the next section.

(iii) The technique used in this section is similar to one used to prove results about probabilities of large deviations. There one is interested in obtaining asymptotics of probabilities that are known to tend to zero. One considers the probabilities of these events with respect to another measure that is chosen in such a way that they do not tend to zero. Asymptotics for the original probabilities are obtained by estimating the Radon-Nikodym derivative of one measure with respect to the other. (See Section 1.9 of Durrett (1996), for example.)

4. Convergence to sets of constellations. Let I be an index set enumerating the collection of all possible structures of constellations (i.e., partitions $A_1, ..., A_k$ of $\{1, ..., N\}$ together with a center a_l in each element A_l of the partition). (Either element of a star of size 2 can be considered to be its center – make any convention about which it is to be.) For $i \in I$, let $\mathcal{C}_i$ be the set of all constellations with structure i. In this section, we will determine initial conditions P with the property that

$$\sum_{i \in I} \mathcal{P}^P\big(P(n) \to \mathcal{C}_i\big) = 1.$$

This will rule out any limiting behavior other than convergence to constellations.

Before beginning, we make a few comments about structure of the proof. The explicit bound on the probability of convergence to $\mathcal{C}_i$ provided by Proposition 3.11 guarantees that if $P(n) \not\to \mathcal{C}_i$, then many products of $p_{j,j'}(n)$ must tend to zero. This is stated in Lemma 4.2. Since $P(n)$ is stochastic, not too many of such products can tend to zero. (See (4.12).) Lemmas 4.4, 4.7 and 4.9 provide the needed links between the "many" that appears in the last two sentences. This gives a contradiction unless $P(n) \to \mathcal{C}_i$ for some i. There are fewer cases to consider if $c_{i,j} > 0$ for all $i \neq j$, so the reader may wish to make that assumption in what follows.

We turn now to the formal proofs. The first ingredient is

$$\lim_{n\to\infty} \mathcal{P}^{P(n)}\big(P(m) \to \mathcal{C}_i\big) = 1_{\{P(n)\to\mathcal{C}_i\}} \text{ a.s.} \tag{4.1}$$

This is a combination of Levy's 0–1 law ((5.8) of Chapter 4 of Durrett (1996)) and the Markov property. When combined with Proposition 3.11 (multiply the inequality in that statement by -1 and exponentiate), it gives the following result.

Lemma 4.2. *Suppose that $\mathcal{C}_i$ has structure $A_1, ..., A_k; a_1, ..., a_k$, and that $c_{j,a_l} > 0$ for all $j \in A_l \backslash \{a_l\}$ and all $1 \le l \le k$. Then*

$$\lim_{n\to\infty} \prod_{l=1}^{k} \prod_{j \in A_l \backslash \{a_l\}} p_{j,a_l}(n) \Bigg(\sum_{j \in A_l} p_{a_l,j}(n) \Bigg) = 0 \quad a.s. \tag{4.3}$$

on the event $\{P(n) \not\to \mathcal{C}_i\}$.

For the next step, we need the following result. The first part will be useful in interchanging the roles of j and a_l in terms that appear in (4.3). The second will be helpful in cases in which some of the $c_{i,j}$'s are zero, when Lemma 4.2 cannot be applied at all.

Lemma 4.4. *(a) For any i, j, if $c_{i,j} > 0$, then*

$$\{p_{i,j}(n) \prod_{(k,l)\in T} p_{k,l}(n) \to 0\} = \{p_{j,i}(n) \prod_{(k,l)\in T} p_{k,l}(n) \to 0\} \quad a.s.$$

for any $T \subset \{1, ..., N\}^2 \backslash \{(k,k) : 1 \le k \le N\}$.

(b) For any i, j, l, if $c_{i,j} = 0$ and $c_{l,i} > 0$, then

$$p_{i,j}(n) p_{l,i}(n) \to 0 \quad a.s. \tag{4.5}$$

Proof. Let $\mathcal{F}_n$ be the σ-algebra generated by the process up to time n. At time n, with probability $\frac{1}{N} p_{i,j}(n)$, the pair (i,j) is chosen. If it is, then

$$p_{j,i}(n+1) = p_{j,i}(n) + c_{i,j}[1 - p_{j,i}(n)] \ge c_{i,j}$$

and

$$p_{k,l}(n+1) \ge (1 - c_{i,j}) p_{k,l}(n)$$

for any $k \ne l$. Therefore, letting $H(n) = \prod_{(k,l)\in T} p_{k,l}(n)$,

$$\begin{aligned} \mathcal{P}\Big(p_{j,i}(n+1) \ge c_{i,j}, H(n+1) \ge \epsilon(1 - c_{i,j})^{|T|} \mid \mathcal{F}_n\Big) \\ \ge \frac{1}{N} p_{i,j}(n) 1_{\{H(n) \ge \epsilon\}}. \end{aligned} \tag{4.6}$$

By the extended Borel-Cantelli Lemma (Corollary 3.2 of Chapter 4 of Durrett (1996)),

$$\{p_{i,j}(n) H(n) \not\to 0\} \subset \{p_{j,i}(n) H(n) \not\to 0\} \quad \text{a.s.}$$

Now interchange the roles of i and j to complete the proof of part (a).

For part (b), note that $c_{i,j} = 0$ implies that $p_{i,j}(n) \downarrow$ in n. By the argument that led to (4.6), we have

$$\mathcal{P}\big(p_{i,j}(n+1) = p_{i,j}(n)(1 - c_{l,i}) \mid \mathcal{F}_n\big) \ge \frac{1}{N} p_{l,i}(n),$$

and hence

$$\{p_{l,i}(n) \not\to 0\} \subset \{p_{i,j}(n+1) = p_{i,j}(n)(1 - c_{l,i}) \text{ i.o.}\} \subset \{p_{i,j}(n) \to 0\},$$

which implies (4.5).

Next we identify the initial states that must be ruled out in order to guarantee convergence to a set of constellations.

Lemma 4.7. *Fix $i \in \{1, ..., N\}$. If $p_{i,j}(0) > 0$ for some j with $c_{i,j} > 0$, then $p_{i,j'}(n) \downarrow 0$ for all j' with $c_{i,j'} = 0$.*

Proof. If $c_{i,j} = 0$, then $p_{i,j}(n) \downarrow$. Let α_j be the (possibly random) limit. We must show $\mathcal{P}\big(\alpha_j = 0 \text{ for all } j : c_{i,j} = 0\big) = 1$. Take a j such that $c_{i,j} = 0$. By Lemma 4.4(b),

$$\{\alpha_j > 0\} \subset \{p_{l,i}(n) \to 0 \text{ for all } l \text{ with } c_{l,i} > 0\} \quad \text{a.s.}$$

But by Lemma 4.4(a),

$$\{p_{l,i}(n) \to 0 \text{ for all } l \text{ with } c_{l,i} > 0\} = \{p_{i,l}(n) \to 0 \text{ for all } l \text{ with } c_{l,i} > 0\} \quad \text{a.s.}$$

Now write

$$1 = \sum_{l=1}^{N} p_{i,l}(n) = \sum_{l:c_{i,l}>0} p_{i,l}(n) + \sum_{l:c_{i,l}=0} p_{i,l}(n). \tag{4.8}$$

On the event $\{\alpha_j > 0\}$, we can let $n \to \infty$ in (4.8) to conclude that

$$1 = \sum_{l:c_{i,l}=0} \alpha_l \le \sum_{l:c_{i,l}=0} p_{i,l}(0) = 1 - \sum_{l:c_{i,l}>0} p_{i,l}(0) < 1 \quad \text{a.s.}$$

This contradiction shows that $\mathcal{P}(\alpha_j > 0) = 0$ as required.

The final bit of preparation is a combinatorial result.

Lemma 4.9. *Suppose S is a finite set, and $f : S \to S$ is a function on S that satisfies $f(x) \ne x$ for every $x \in S$. Then there is a partition $A_1, ..., A_k$ of S and a collection of centers $a_l \in A_l$ so that*

$$\cup_{l=1}^{k} \big\{\{i, a_l\} : i \in A_l \backslash \{a_l\}\big\} \subset \big\{\{i, f(i)\} : i \in S\big\}$$

Proof. The proof is by induction on the size of the range $\mathcal{R}$ of f. Note that $\mathcal{R}$ must consist of at least two points. For the basis step, suppose that $\mathcal{R} = \{a, b\}$. Let $A = \{x \in S : f(x) = a\}$ and $B = \{x \in S : f(x) = b\}$. Then $a \in B$ and $b \in A$. If $A = \{b\}$ and $B = \{a\}$, take $k = 1, A_1 = \{a, b\}, a_1 = a$. If $A = \{b\}$ and $|B| > 1$, let $k = 1, A_1 = S, a_1 = b$. If $B = \{a\}$ and $|A| > 1$, let $k = 1, A_1 = S, a_1 = a$. Finally, if $|A| > 1$ and $|B| > 1$, take $k = 2, A_1 = (B \cup \{b\}) \backslash \{a\}, a_1 = b, A_2 = (A \cup \{a\}) \backslash \{b\}, a_2 = a$.

Turning to the induction step, suppose that $\mathcal{R} = \{b_1, ..., b_m\}$ for some $m > 2$, and let $B_i = \{x \in S : f(x) = b_i\}$. There are three cases to consider:

(i) If there is an $1 \le i \le m$ so that

$$f(b_j) \ne b_i \text{ for every } 1 \le j \le m, \tag{4.10}$$

then take $A_1 = \{b_i\} \cup B_i$ and $a_1 = b_i$ for that i. Note that $B_i \ne \emptyset$, since b_i is in the range of f. Let $S' = S \backslash A_1$. Then $f : S' \to S'$, and the range of f restricted to

S' is a subset of $\{b_j : j \neq i\}$, so that by the induction hypothesis, we may further partition S' in such a way that the required properties hold. The desired partition of S is obtained by adjoining A_1, a_1 to the partition of S'.

(ii) If there is no i satisfying (4.10), then the restriction of f to $\{b_1, ..., b_m\}$ is a permutation of this set. Write this permutation as a product of cycles. If there are two or more cycles, then letting $(b_{i_1}, ..., b_{i_l})$ be one of the cycles, write $S = S_1 \cup S_2$, where $S_1 = \cup_{j=1}^{l} B_{i_j}$ and $S_2 = S \backslash S_1$. Then f maps S_i to S_i for $i = 1, 2$, so that each may be partitioned appropriately by the induction hypothesis. The desired partition of S is obtained by combining these two partitions.

(iii) If the permutation in (ii) consists of a single cycle, then we may assume by relabeling that the cycle is $(b_1, b_2, ..., b_m)$. The required partition is then constructed as follows: For any i such that $|B_i| > 1$, we let a member of the partition be $(B_i \cup \{b_i\}) \backslash \{b_{i-1}\}$ (with the convention that $b_0 = b_m$). The corresponding center is taken to be b_i. What remains in S is a set of b_i's, which in the cycle order, breaks down into connected intervals of the form $b_i, b_{i+1}, ..., b_j$. For each such set, if $j - i$ is even, add b_j to the element of the partition that has been created so far with center b_{j+1}. The remaining interval, $b_i, b_{i+1}, ..., b_{j-1}$ or $b_i, b_{i+1}, ..., b_j$, consists of an even number of points, which can be partitioned into a set of adjacent pairs, with either element of the pair taken to be its center. This leaves only the case in which $B_i = \{b_{i-1}\}$ for each i. In this case, take the partition to be $\big\{\{b_1, b_2\}, ..., \{b_{m-1}, b_m\}\big\}$ if m is even, and $\big\{\{b_1, b_2\}, ..., \{b_{m-4}, b_{m-3}\}, \{b_{m-2}, b_{m-1}, b_m\}\big\}$ if m is odd. Any member of a pair can be taken to be its center. In the case of the triple $\{b_{m-2}, b_{m-1}, b_m\}$, take the center to be b_{m-1}.

We are now in a position to prove the main result in this section.

Proposition 4.11. *Suppose that for each $i \in \{1, ..., N\}$, $p_{i,j} > 0$ for some j with $c_{i,j} > 0$. Then*

$$\sum_{l \in I} \mathcal{P}^P \big(P(n) \to \mathcal{C}_l\big) = 1.$$

Proof. Since $P(n)$ is stochastic for each n,

$$1 = \prod_{i=1}^{N} \sum_{j=1}^{N} p_{i,j}(n). \tag{4.12}$$

Multiplying out this expression, we see that a typical term is of the form

$$\prod_{i=1}^{N} p_{i,j_i}(n), \tag{4.13}$$

where $j_i \in \{1, ..., N\}$ and $j_i \neq i$ for each i. Let $A_1, ..., A_k$ and $a_1, ..., a_k$ be the partition of $\{1, ..., N\}$ and corresponding centers given in Lemma 4.9 for the function $f(i) = j_i$. Then (4.13) is bounded above by

$$\prod_{l=1}^{k} \prod_{i \in A_l \backslash \{a_l\}} [p_{i,a_l}(n) + p_{a_l,i}(n)] \tag{4.14}$$

If $c_{i,a_l} = 0$ for some $1 \leq l \leq k$ and some $i \in A_l \backslash \{a_l\}$, then (4.14) tends to zero as $n \to \infty$ by Lemma 4.7. Otherwise, all these $c_{i,j}$ are strictly positive, and hence by Lemma 4.2,

$$\lim_{n\to\infty} \prod_{l=1}^{k} \prod_{j \in A_l \backslash \{a_l\}} p_{j,a_l}(n) \Big(\sum_{j \in A_l} p_{a_l,j}(n) \Big) = 0 \quad \text{a.s.} \tag{4.15}$$

on the event $\{P(n) \not\to \mathcal{C}\}$, where $\mathcal{C}$ is the set of all constellations with structure $A_1, ..., A_k; a_1, ..., a_k$. After multiplying out the sums in (4.15) a typical term is of the form $X_n Y_n$, where

$$X_n = \prod_{l=1}^{k} \prod_{j \in A_l \backslash \{a_l\}} p_{j,a_l}(n) \quad \text{and} \quad Y_n = \prod_{l=1}^{k} p_{a_l,j_l}(n),$$

where $j_l \in A_l \backslash \{a_l\}$. (These are not the j_i's that appear in (4.13).) Now,

$$\mathcal{P}\big(p_{a_l,j_l}(n+k) \geq c_{j_l,a_l} \;\forall\, 1 \leq l \leq k \text{ and } X_{n+k} = X_n \mid \mathcal{F}_n\big) \geq \prod_{l=1}^{k} \frac{p_{j_l,a_l}(n)}{N} \geq N^{-k} X_n,$$

since the right side is the probability that the choices made at times $n, ..., n+k-1$ are $(j_1, a_1), ..., (j_k, a_k)$, and if these choices are made, then $p_{a_l,j_l}(n+k) \geq c_{j_l,a_l}$ for $1 \leq l \leq k$, and since only the rows with indexes $a_1, ..., a_k$ have been altered, $X_{n+k} = X_n$. Therefore, setting $\epsilon = \prod_l c_{j_l,a_l} > 0$, we have

$$\mathcal{P}\big(Y_{n+k} X_{n+k} \geq \epsilon\delta \mid \mathcal{F}_n\big) \geq N^{-k} X_n 1_{\{X_n \geq \delta\}} \quad \text{a.s.}$$

for any $\delta > 0$. By the extended Borel-Cantelli Lemma, it follows that

$$\{X_n \not\to 0\} \subset \{X_n Y_n \not\to 0\}.$$

But by (4.15) $X_n Y_n \to 0$ a.s. on the event $\{P(n) \not\to \mathcal{C}\}$. Therefore, $X_n \to 0$ a.s. on the event $\{P(n) \not\to \mathcal{C}\}$. Using Lemma 4.4(a) repeatedly, it follows that (4.14), and hence (4.13), tends to zero as $n \to \infty$.

We have now shown that on the event $\cap_{l\in I}\{P(n) \not\to \mathcal{C}_l\}$, the right side of (4.12) tends to zero a.s. Since the left side is 1, it follows that

$$\mathcal{P}\big(\cap_{l\in I} \{P(n) \not\to \mathcal{C}_l\}\big) = 0$$

as required.

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