T. Liggett Mathematics 31B – Second Midterm Solutions May 15, 2009

(15) 1. In each case, determine whether the integral converges. If it does, compute its value.

(a) $\int_0^\infty \frac{x}{1+x^2}dx$ diverges. Compare with $\int_1^\infty \frac{1}{2x}dx$, or write

$$\int_0^R \frac{x}{1+x^2}dx = \frac{1}{2}\ln(1+R^2) \to \infty \text{ as } R \to \infty.$$

(b) $\int_0^9 \frac{1}{\sqrt{9-x}}dx$ converges to 6, since

$$\lim_{R\to 9-}\int_0^R \frac{1}{\sqrt{9-x}}dx = \lim_{R\to 9-}[6-2\sqrt{9-R}] = 6.$$

(10) 2. Evaluate

$$\int_0^{\pi/2} x \sin 4x dx$$

Integrate by parts with $u = x, v' = \sin 4x, u' = 1, v = -\frac{1}{4}\cos 4x$:

$$\int_0^{\pi/2} x\sin 4x dx = -\frac{x}{4}\cos 4x\Big|_0^{\pi/2} + \int_0^{\pi/2}\frac{1}{4}\cos 4x dx = -\frac{\pi}{8}.$$

(10) 3. Find the arc length of the graph of $y = \frac{x^4}{16} + \frac{1}{2x^2}$ from $x = 2$ to $x = 4$.

$$\text{Arc length } = \frac{1}{4}\int_2^4 \frac{x^6+4}{x^3}dx = \frac{483}{32}.$$

(15) 4. Evaluate:

(a) $$\int \frac{x^2}{\sqrt{9-x^2}}dx$$

Use $x = 3\sin\theta, dx = 3\cos\theta d\theta$:

$$\begin{aligned}\int \frac{x^2}{\sqrt{9-x^2}}dx &= \int \frac{9\sin^2\theta}{3\cos\theta}3\cos\theta d\theta = 9\int \sin^2\theta d\theta \\ &= -\frac{9}{2}\sin\theta\cos\theta + \frac{9}{2}\theta + C \\ &= -\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}(x/3) + C.\end{aligned}$$

(b)
$$\begin{aligned}\int_0^{\pi/2} \sin^2 x\cos^3 x dx &= \int_0^{\pi/2} \sin^2 x(1-\sin^2 x)\cos x dx \\ &= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x\Big|_0^{\pi/2} = \frac{2}{15}.\end{aligned}$$

(10) 5. Find the Simpson Rule approximation to $\int_1^2 \frac{1}{x}dx$ for $n = 4$. (There is no need to add up the fractions in your answer.)

$$\frac{1}{3}\frac{1}{4}\left(1+4\cdot\frac{4}{5}+2\cdot\frac{2}{3}+4\cdot\frac{4}{7}+\frac{1}{2}\right)=\frac{1747}{2520}.$$

(15) 6. (a) Find the fourth degree Maclaurin polynomial (i.e., Taylor polynomial about 0) $T_4(x)$ for $f(x) = \ln(1+x)$.

$$T_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

(b) What is the error bound for $|T_4(1) - \ln 2|$?

$f^{(5)}(x) = 24/(1+x)^5$, so $K = 24$. The error bound is $24/5! = 1/5$.

(10) 7. Evaluate

$$\int \frac{11x+17}{2x^2+7x-4}dx = \int \left(\frac{3}{x+4}+\frac{5}{2x-1}\right)dx = 3\ln|x+4|+\frac{5}{2}\ln|2x-1|+C.$$

(15) 8. (a) For a sequence a_n, define: $\lim_{n\to\infty} a_n = L$.

$\forall\epsilon > 0\ \exists N$ such that $n > N$ implies $|a_n - L| < \epsilon$.

(b) State the Squeeze Theorem for Sequences.

If $a_n \le b_n \le c_n$ for all but finitely many n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

(c) Prove the Squeeze Theorem for Sequences.

Given $\epsilon > 0$, there exist N_1 and N_2 so that $n > N_1$ implies $|a_n - L| < \epsilon$ and $n > N_2$ implies $|c_n - L| < \epsilon$. Let $N = \max(N_1, N_2)$. Then $n > N$ implies $|a_n - L| < \epsilon$ and $|c_n - L| < \epsilon$, and therefore implies that

$$-\epsilon < a_n - L \le b_n - L \le c_n - L < \epsilon,$$

i.e. that $|b_n - L| < \epsilon$.