T. Liggett Mathematics 31B – Final Exam June 7, 2009

(11) 1. Consider the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n2^n}.$$

(a) Find the radius of convergence. Answer: 2.

(b) Find the interval of convergence. Answer: $[1,5)$.

(14) 2. (a) Evaluate

$$\sum_{n=0}^{\infty} \frac{1+2^n}{6^n} = \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{6}{5} + \frac{3}{2} = \frac{27}{10}.$$

(b) Use partial fractions to evaluate

$$\sum_{n=1}^{\infty} \frac{1}{9n^2-3n-2} = \frac{1}{3}\sum_{n=1}^{\infty} \left(\frac{1}{3n-2} - \frac{1}{3n+1}\right) = \frac{1}{3}\left(1 - \frac{1}{4} + \frac{1}{4} - \frac{1}{7} + \cdots\right) = \frac{1}{3}.$$

(15) 3. Find a solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ of the differential equation $y'' - 2xy' - 2y = 0$ satisfying $y(0) = 0, y'(0) = 1$ as follows:

(a) What are a_0 and a_1? Answer: $a_0 = y(0) = 0, a_1 = y'(0) = 1$.

(b) Find a recursion that allows you to compute each a_n in terms of the previous ones.

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2},$$

so the differential equation becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2\sum_{n=0}^{\infty} n a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

Equating coefficients of x^n leads to

$$a_{n+2} = \frac{2}{n+2} a_n.$$

(c) Find a_2 and a_3. Answer: $a_2 = a_0 = 0, a_3 = (2/3)a_1 = 2/3$.

(d) Find a_n for general n. Answer: $a_{2n} = 0$, and $a_{2n+1} = \frac{2}{2n+1} a_{2n-1}$, so

$$a_{2n+1} = \frac{2^n}{(2n+1)(2n-1)\cdots 3\cdot 1} = \frac{4^n n!}{(2n+1)!}.$$

(10) 4. Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

(a) Does it converge or diverge? Answer: It converges.

(b) Explain your answer in (a). Answer: Use the integral test, with

$$f(x) = \frac{1}{x(\ln x)^2}; \quad \int_2^\infty f(x)dx = \int_{\ln 2}^\infty \frac{du}{u^2} < \infty.$$

(15) 5. Let $f(x) = 1/(1-x)$.

(a) Find the nth Taylor polynomial $T_n(x)$ of f about 0.

$$f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}, \text{ so } f^{(k)}(0) = k!, \text{ and } T_n(x) = \sum_{k=0}^{n} x^k.$$

(b) Find a good upper bound (depending on x) for $|T_n(x) - f(x)|$, valid for $|x| \le \frac{1}{2}$.

Use the error bound with $K = (n+1)!2^{n+2}$ to get $|T_n(x) - f(x)| \le 2(2|x|)^{n+1}$.

(c) Find the Taylor series for f about 0. From part (a), it is

$$\sum_{k=0}^{\infty} x^k.$$

(d) Use the result of part (b) to show that the Taylor series for f about 0 converges to f for $|x| < \frac{1}{2}$.

$$\lim_{n\to\infty} |T_n(x) - f(x)| \le \lim_{n\to\infty} (2|x|)^{n+1} = 0.$$

(10) 6. Find the equation of the tangent line to $y = \ln(\sin x)$ at $x = \pi/4$.

$y' = \cot x = 1$ at $x = \pi/4$. $y = -\ln 2/2$ at $x = \pi/4$. So, the equation is $y + \ln 2/2 = x - \pi/4$.

(13) 7. Evaluate:

(a) $$\lim_{x\to 0} \ln x \tan^{-1}(2x)$$

Use L'Hopital:

$$\lim_{x\to 0} \ln x \tan^{-1}(2x) = \lim_{x\to 0} \left[-\frac{2x(\ln x)^2}{1+4x^2} \right] = 0.$$

(b) $$\int \frac{\tan^{-1} x}{1+x^2} dx = \frac{1}{2}(\tan^{-1} x)^2 + C.$$

(12) 8. Evaluate:

(a) $\dfrac{d}{dx}e^{(x^2+2x+3)^2} = 4(1+x)(3+2x+x^2)e^{(x^2+2x+3)^2}$

(b)
$$\int \frac{e^{2x}-e^{4x}}{e^x}dx = \int e^x dx - \int e^{3x}dx = e^x - \frac{1}{3}e^{3x} + C.$$

(12) 9. Decide whether each of the following statements is true or false, and put T or F in the box. (Scoring: for each one, 2 points for the right answer, -1 point for the wrong answer, and 0 points for no answer.)

(a) If a_n is bounded, then it converges. F

(b) The sequence $a_n = (-1)^n/(n^2+1)$ is monotonic. F

(c) Every convergent sequence is bounded. T

(d) Every decreasing sequence converges. F

(e) The sequence $a_n = \sqrt{n+1} - \sqrt{n}$ is decreasing. T

(f) If $a_n + b_n$ converges, then a_n converges. F

(13) 10. Evaluate
$$\int_0^\pi \sin^4 x \cos^4 x dx = \frac{3\pi}{128}.$$

(12) 11. Find
$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}.$$

Note that for $1 \le k \le n$,
$$\frac{1}{\sqrt{n(n+1)}} \le \frac{1}{\sqrt{n^2+k}} \le \frac{1}{\sqrt{n^2+1}},$$

so that

$$\frac{n}{\sqrt{n(n+1)}} \le \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \le \frac{n}{\sqrt{n^2+1}}.$$

The sequences on the left and right each tend to 1 as $n \to \infty$, so the limit of the middle sequence is 1 also by the squeeze theorem.

(13) 12. Use integration by parts (twice) to evaluate

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C.$$

In problems 13 and 14, decide whether the series (1) converges absolutely, (2) converges conditionally, or (3) diverges, and put the corresponding number (1,2 or 3) in the box provided. No explanation is required. (Scoring: in each case, 3 points for the right answer, -1 point for the wrong answer and 0 points for no answer.)

(15) 13.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ 3

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ 1

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$ 1

(d) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$ 2

(e) $\displaystyle 1 + \frac{1\cdot 2}{1\cdot 3} + \frac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5} + \frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7} + \cdots$ 1

(15) 14.

(a) $\displaystyle 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ 3

(b) $$\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n^2}\right)$$ 1

(c) $$\sum_{n=1}^{\infty} (-1)^n \left(\frac{n-1}{n}\right)^n$$ 3

(d) $$1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots$$ 2

(e) $$\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$$ 1

(10) 15. (a) Prove: If $\sum_n a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Letting S_n be the partial sums and S be the sum, $\lim_n a_n = \lim_n (S_n - S_{n-1}) = S - S = 0$.

(b) Give an example to show that $\lim_{n\to\infty} a_n = 0$ does not imply that $\sum_n a_n$ converges. There is no need to show that your example has the properties you claim it has.

The harmonic series.

(10) 16. (a) Prove: If $\sum_n a_n$ converges absolutely, then $\sum_n a_n$ converges.

Using $0 \le a_n + |a_n| \le 2|a_n|$, we see that $\sum_n (a_n + |a_n|)$ converges. Since $\sum_n a_n$ is the difference of the two convergence series $\sum_n (a_n + |a_n|)$ and $\sum_n |a_n|$, it converges.

(b) Give an example to show that $\sum_n a_n$ converges does not imply that $\sum_n a_n$ converges absolutely. There is no need to show that your example has the properties you claim it has.

The alternating harmonic series.

Scratch paper – If you detach it, do NOT turn it in!

You may find the following formulas useful:

Error bound for Simpson's rule: If K_4 satisfies $|f^{(4)}(x)| \le K_4$ for all $x \in [a, b]$, then

$$\text{Error } (S_N) \le \frac{K_4(b-a)^5}{180N^4}.$$

Reduction formulas:

$$\int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x dx$$

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

Arc length of the graph of $y = f(x)$ over $[a, b]$:

$$\int_a^b \sqrt{1+[f'(x)]^2}dx.$$

Surface area of the surface obtained by rotating the graph of $f(x)$ for $a \le x \le b$ about the x-axis:

$$2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}dx.$$

Error bound for the n^{th} Taylor polynomial: If $|f^{(n+1)}(u)| \le K$ for all u between a and x, then

$$|T_n(x) - f(x)| \le K\frac{|x-a|^{n+1}}{(n+1)!}.$$

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}, \quad \frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}.$$