Mathematics 171 – HW1 – Due Thursday, April 7, 2011.

Problems 1,2,3,4,5 on pages 88-89, plus the following:

A. Let y, z be distinct states. Show that $T = \min\{n \geq T_y : X_n = z\}$ is a stopping time.

B. Consider two urns A and B containing a total of N balls. Let X_n be the number of balls in urn A at time n. If $X_n = k$, an urn is chosen with probabilities $\frac{k}{N}$ and $1 - \frac{k}{N}$ respectively, and a ball is chosen uniformly from all N balls. The chosen ball is placed in the chosen urn. Find the transition probabilities for this Markov chain.

C. Recall that a stochastic matrix is one that has non-negative entries and row sums equal to 1. Every such matrix corresponds to a Markov chain. Not every stochastic matrix can be the two-step transition matrix for a Markov chain. Show that a 2×2 stochastic matrix is the two-step transition matrix for a Markov chain if and only if the sum of its diagonal entries is at least one.

D. Suppose that $X_1, X_2, \dots$ are random variables taking the values 0 and 1, and satisfying

$$P(X_n = 1 \mid X_1 = i_1, \dots, X_{n-1} = i_{n-1}) \geq \epsilon$$

for all $n \geq 1$ and all choices of $i_1, \dots, i_{n-1} \in \{0, 1\}$. Show that if $\epsilon > 0$, then

(a) $P(X_n = 1 \text{ for some } n) = 1$,

and

(b) $P(X_n = 1 \text{ for infinitely many } n) = 1$.

(Suggestion: Look at the complementary events.)