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1. The continuous time renewal chain X_t has state space $S = \{0, 1, 2, ...\}$ and transition rates (i.e., $Q-$matrix) given by

$$q(n, n-1) = \delta_n \quad \text{and} \quad q(0, n) = q_n, \quad n \geq 1.$$

All other $q(n, m) = 0$ for $n \neq m$. Assume that all the δ_n's and q_n's are strictly positive, and the δ_n's are uniformly bounded. Determine when the chain is positive recurrent, and when it is, compute its stationary distribution.

Solution: The equations for π are

$$\pi(n+1)\delta_{n+1} + \pi(0)q_n = \pi(n)\delta_n, \quad n \geq 1.$$

Iterating this gives

$$\pi(n)\delta_n = \pi(0)\sum_{k=n}^{\infty} q_k.$$

Therefore, the chain is positive recurrent if and only if

$$\sum_{n=1}^{\infty} \frac{1}{\delta_n} \sum_{k=n}^{\infty} q_k < \infty.$$

2. Suppose orders to a factory arrive according to a rate 1 Poisson process $N(t)$. Each order is independently of type a, b or c with probability $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{6}$ respectively.

(a) Given that there are exactly three arrivals in the interval $[0, 1]$, what is the probability that there is one of each type?

Solution: 6(1/36)=1/6.

(b) What is the expected number of arrivals of orders of type a in the interval $[0, 1]$?

Solution: The type a orders arrive according to a rate $\frac{1}{2}$ Poisson process, so the answer is $\frac{1}{2}$.

(c) Let X be the number of arrivals of type a or b during the interval $[0, 2]$ and Y be the number of arrivals of type b or c during the interval $[1, 3]$. Find EXY.

Solution: Write $X = X_1 + X_2 + Z$ and $Y = Y_1 + Y_2 + Z$, where, for example, Z is the number of arrivals of type b in the interval $[1, 2]$. These are independent Poisson random variables with parameters 1 for X_1, and $\frac{1}{3}$ for the others. Therefore, since $EZ^2 = 4/9$,

$$EXY = E(X_1 + X_2 + Z)(Y_1 + Y_2 + Z) = 3(1/3) + 5(1/9) + 4/9 = 2.$$

3. Consider two machines that are maintained by a single repairman. Machine i functions for an exponentially distributed amount of time with rate λ_i. The repair times for each machine are exponential with rate μ_i. They are repaired in the order in which they fail. At any given time, there are five possible states for the system: a: both are working; b_i: machine i is working and the other is being repaired; c_i: neither machine is working and machine i is being repaired. Fill in the values of the $Q-$matrix below.

4. Suppose $Q = (q_{i,j})$ is the $Q-$matrix for a continuous time Markov chain, λ_i are the rates of the exponential holding times, and $p_{i,j}$ are the transition probabilities for the embedded discrete time chain. What are the equations that relate these quantities?

Solution: $\lambda_i = -q_{i,i}$ and $p_{i,j} = q_{i,j}/\lambda_i$ for $i \neq j$.

5. Consider a continuous time Markov chain with $Q-$matrix $Q = (q_{i,j})$ and transition probabilities $p_t(i, j)$.

(a) State the Chapman-Kolmogorov equations.

Solution:

$$p_{t+s}(i, j) = \sum_k p_t(i, k)p_s(k, j).$$

(b) State either the Kolmogorov backward or forward equations. Be sure to say which you are stating.

Solution: See pages 165-166 of the text.

6. For each statement below, say whether it is true or false. No explanation is necessary. (Scoring: $+3$ if correct, -1 if incorrect, 0 if no answer.)

(a) A continuous time Markov chain is positive recurrent if and only if its embedded discrete time chain is positive recurrent.

Solution: False.

(b) A continuous time Markov chain is recurrent if and only if its embedded discrete time chain is recurrent.

Solution: True.

(c) Suppose a continuous time Markov chain with $Q-$matrix $Q = (q(i,j))$ satifies $1 \leq \lambda_i = -q(i,i) \leq 2$. Then it is positive recurrent if and only if its embedded discrete time chain is positive recurrent.

Solution: True.

(d) If M_n is a martingale and τ is a stopping time, then $EM_\tau = EM_0$.

Solution: False.

(e) If $P_x(\tau_x < \infty) = 1$, then $E\tau_x < \infty$.

Solution: False.

(f) In an inhomogeneous Poisson process, the time of the first arrival is exponentially distributed.

Solution: False.

7. Consider the continuous time Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ and the following $Q-$matrix:

(a) What is the expected time spent at state 1 each time the chain visits state 1?

Solution: Since the holding time at state 1 has rate 6, the expected time spent at each visit is $\frac{1}{6}$.

(b) Find the stationary distribution of the chain.

Solution: The detailed balance condition is

$$\pi(1)2 = \pi(2)1, \pi(3)2 = \pi(2)3, \pi(4)1 = \pi(1)4, \pi(5)2 = \pi(2)5, \pi(4)3 = \pi(3)4, \pi(5)4 = \pi(4)5.$$

This is satisfied by $\pi(i) = i$. Normalizing gives $\pi(i) = \frac{i}{15}$.

(c) Does the stationary distribution satisfy the detailed balance condition? Explain.

Solution: Yes; see above.

(d) What is the long time proportion of time the chain spends in $\{1, 2\}$?

Solution: $\pi(1) + \pi(2) = \frac{1}{5}$.

8. Consider the discrete time random walk on the following graph – at each time, the chain moves to a neighbor chosen at random.

(a) Find $E_5\tau_1$.

Solution: Let $f(k) = E_k\tau_1$. By symmetry, $f(2) = f(4)$ and $f(3) = f(6)$. Let $x = f(2) = f(4)$, $y = f(3) = f(6)$ and $z = f(5)$. Then

$$z = 1 + \frac{1}{2}(x + y), \quad y = 1 + \frac{1}{2}(x + z), \quad x = 1 + \frac{1}{4}(y + z).$$

Solving gives $x = 8$ and $y = z = 10$.

(b) Find $P_5(\tau_1 < \tau_3)$.

Solution: Let $g(k) = P_k(\tau_1 < \tau_3)$. Then $g(1) = 1, g(3) = 0$, and by symmetry, $g(2) = g(6) = \frac{1}{2}$ and $g(4) + g(5) = 1$. Using this information, $g(5) = \frac{1}{4} + \frac{1}{4}g(4)$ and $g(4) = \frac{1}{2} + \frac{1}{4}g(5)$. Solving gives $g(4) = \frac{3}{5}$ and $g(5) = \frac{2}{5}$.

9. Suppose $N(A), A \subset R^2$ is a two-dimensional spatial Poisson process of rate 1. Let $f(x)$ be a positive continuous function on R^1, and define

$$M(t) = N(\{(x, y) \in R^2 : 0 \le x \le t \text{ and } |y| \le f(x)\}).$$

(a) What is the distribution of $M(1)$?

Solution: Poisson with parameter $2\int_0^1 f(x)dx$.

(b) What is the distribution of $\tau = \min\{t \geq 0 : M(t) \geq 1\}$?

Solution:

$$P(\tau > u) = P(M(u) = 0) = e^{-2\int_0^u f(x)dx},$$

so τ has density

$$2f(u)e^{-2\int_0^u f(x)dx}.$$

10. Jobs arrive at a single server queue according to a Poisson process of rate λ. The time it takes the server to handle each is exponentially distributed with parameter μ. Let X_t be the number of jobs in the system (either being handled or waiting for service) at time t.

(a) Find the $Q-$ matrix for this chain.

Solution: $q(n, n+1) = \lambda, q(n, n-1) = \mu, q(n, n) = -(\lambda + \mu)$ for $n \geq 1$ and $q(0, 1) = \lambda, q(0, 0) = -\lambda$.

(b) Find conditions on λ and μ for the chain to be positive recurrent, and compute the stationary distribution in that case.

Solution: The detailed balance equations are $\pi(n)\lambda = \pi(n+1)\mu$ for $n \geq 0$. Solving gives $\pi(n) = \pi(0)(\lambda/\mu)^n$. So, the chain is positive recurrent iff $\lambda < \mu$, and then

$$\pi(n) = \frac{\mu - \lambda}{\mu}\left(\frac{\lambda}{\mu}\right)^n.$$

(c) What proportion of the time is the system empty?

Solution: $\pi(0) = \frac{\mu-\lambda}{\mu}$.

11. Let X_n be an asymmetric random walk on the integers, with

$$p(k, k+1) = p, \quad p(k, k-1) = q = 1 - p, \quad (p > q).$$

(a) Find a positive number $\theta \neq 1$ for which θ^{X_n} is a martingale.

Solution: $E(\theta^{X_{n+1}} \mid \mathcal{F}_n) = \theta^{X_n}(\theta p + \theta^{-1}q)$. So, the value of θ that makes this a martingale is $\theta = q/p$.

(b) Use the martingale in part (a) to compute $P_k(\tau_N < \tau_0)$ for $0 < k < N$.

Solution: Let $\tau = \min(\tau_0, \tau_N)$. Then

$$\theta^k = E_k \theta^{X_\tau} = \theta^N P_k(\tau_N < \tau_0) + 1 - P_k(\tau_N < \tau_0).$$

Solving gives

$$P_k(\tau_N < \tau_0) = \frac{1 - \theta^k}{1 - \theta^N} = p^{N-k} \frac{p^k - q^k}{p^N - q^N}.$$