Suppose $X_n$ is $B(n,p)$.

(a) Find the moment generating function $M_n(s)$ of $(X_n - np)/\sqrt{np(1-p)}$.

Write $q = 1-p$. The MGF of $X_n$ is $(pe^s + q)^n$, since $X_n$ can be written as the sum of $n$ independent Bernoulli’s with parameter $p$, and these have MGF $pe^s + q$. Therefore,

$$M_n(s) = E \exp \left\{ \frac{s}{\sqrt{npq}} [pe^{s/\sqrt{npq}} + q] \right\} = [pe^{s/\sqrt{np}} + qe^{-s/\sqrt{p/npq}}]^n.$$ 

(b) Compute the limit

$$\lim_{n \to \infty} M_n(s),$$
directly, without using the central limit theorem.

We want to write $M_n(s)$ in the form $(1 + a_n/n)^n$. Solving for $a_n$ gives

$$a_n = n [pe^{s/\sqrt{np}} + qe^{-s/\sqrt{p/npq}} - 1].$$

Recalling that the expansion of the exponential is $e^x = 1 + x + x^2/2 + \cdots$ suggests that this should be rewritten in the form

$$a_n = np [e^{s\sqrt{q/np}} - 1 - s\sqrt{q/np}] + nq [e^{-s\sqrt{p/npq}} - 1 + s\sqrt{p/npq}].$$

Since

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}$$

(by applying L’Hopital’s rule twice),

$$\lim_{n \to \infty} a_n = \frac{s^2}{2}.$$ 

So,

$$\lim_{n \to \infty} M_n(s) = e^{s^2/2},$$

which is the mgf of the $N(0, 1)$ distribution.

Suppose $X_n$ is Poisson with parameter $n$.

(a) Find the moment generating function $M_n(s)$ of $(X_n - n)/\sqrt{n}$.

Since the MGF of the Poisson with parameter $n$ is $e^{n(e^s - 1)}$,

$$M_n(s) = e^{-s\sqrt{n} + n(e^{s/\sqrt{n}} - 1)}.$$ 

(b) Compute the limit

$$\lim_{n \to \infty} M_n(s),$$
directly, without using the central limit theorem.

Taking logs gives

$$\log M_n(s) = n \left[ e^{s/\sqrt{n}} - \frac{s}{\sqrt{n}} - 1 \right].$$
so using (1) again gives

\[ \lim_{n \to \infty} \log M_n(s) = \frac{s^2}{2}, \]

so that

\[ \lim_{n \to \infty} M_n(s) = e^{s^2/2} \]

as in Problem \( F_4 \).

\( G_2. \) Suppose the random variables \( X_n \) satisfy \( EX_n = 0, EX_n^2 \leq 1, \) and \( \text{Cov}(X_n, X_m) \leq 0 \) for \( n \neq m \). Show that

\[ \frac{S_n}{n} = \frac{X_1 + \cdots + X_n}{n} \]

converges to 0 in probability.

**Solution:** The proof follows the proof of the WLLN under a second moment assumption: By Chebyshev,

\[ P(|S_n/n| \geq \epsilon) \leq \frac{1}{\epsilon^2 n^2} \text{var}(S_n). \]

But

\[ \text{var}(S_n) = \sum_{i,j=1}^{n} \text{cov}(X_i, X_j) \leq \sum_{i=1}^{n} \text{var}(X_i) \leq n. \]

Combining these gives

\[ P(|S_n/n| \geq \epsilon) \leq \frac{1}{\epsilon^2 n}, \]

which tends to zero as \( n \to \infty \).

\( G_3. \) Show that in each of the cases (a), (c), and (d) of Problem 5 on page 288, the sequence actually converges a.s.

**Solution:** For (a): \( EY_n^2 = \frac{1}{3n^2} \), so \( \sum_n EY_n^2 < \infty \). Therefore \( Y_n^2 \to 0 \) a.s., so \( Y_n \to 0 \) a.s. For (c), \( E|Y_n| = \frac{1}{2n} \), so \( \sum_n E|Y_n| < \infty \). Therefore \( Y_n \to 0 \) a.s. For (d), there are two possible approaches: One is to show that \( E(1 - Y_n)^2 = \frac{8}{(n+1)(n+2)} \), and proceed as in the other cases. The other is to note that \( Y_n \) is nondecreasing in \( n \) and is bounded above by 1. Therefore, \( Y = \lim_{n \to \infty} Y_n \) exists for every \( \omega \), and satisfies \( Y \leq 1 \). To show that \( Y = 1 \) a.s., take \( 0 < \epsilon < 1 \) and write

\[ P(Y \leq 1 - \epsilon) \leq P(Y_n \leq 1 - \epsilon) = (1 - (\epsilon/2))^n, \]

which tends to zero as \( n \to \infty \). Therefore \( P(Y < 1) = \lim_{\epsilon \downarrow 0} P(Y \leq 1 - \epsilon) = 0. \)
Suppose each $X_n$ takes the values $\pm 1$ with probability $\frac{1}{2}$ each. Show that the random series
\[ \sum_{n=1}^{\infty} \frac{X_n}{n^p} \]
converges a.s. (which means that the partial sums converge a.s.) if $p > 1$.

**Solution:** The series converges absolutely, since
\[ \sum_{n=1}^{\infty} \frac{|X_n|}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty. \]
Now use the fact that absolute convergence of a series implies convergence.

Suppose $X_n$ are i.i.d. non-negative random variables.

(a) Show that $\frac{X_n}{n} \to 0$ in probability with no further assumptions. (You did this before in case they are uniformly distributed on $[-1, 1]$.)

**Solution:**
\[ P(X_n/n > \epsilon) = P(X_1 > n\epsilon) \to 0 \]
as $n \to \infty$.

Consider now two cases: (i) $EX < \infty$ and (ii) $EX = \infty$. Recall that the series
\[ \sum_k P(X_1 > k) = \sum_k P(X_k > k) \]
converges in case (i) and diverges in case (ii). (See Problem 3 on page 184. This gives the statement in terms of integrals rather than sums, but there is no real difference.)

(b) Express $P(X_k \leq k$ for all $k \geq n)$ in terms of the probabilities $P(X_k > k)$.

**Solution:**
\[ \prod_{k=n}^{\infty} [1 - P(X_1 > k)]. \]

(c) Show that
\[ \lim_{n \to \infty} P(X_k \leq k$ for all $k \geq n) = 1 \]
in case (i) and $P(X_k \leq k$ for all $k \geq n) = 0$ for all $n$ in case (ii). (Suggestion: take logs.)
Solution: Since \( \lim_{x \to 0} \frac{\log(1+x)}{x} = 1 \) by L’Hospital, there is an \( \epsilon > 0 \) so that
\[
\frac{x}{2} \leq |\log(1-x)| \leq \frac{3x}{2} \quad \text{for } 0 \leq x \leq \epsilon.
\]
Therefore, for sufficiently large \( k \),
\[
\frac{1}{2}P(X_1 > k) \leq |\log[1 - P(X_1 > k)]| \leq \frac{3}{2}P(X_1 > k).
\]
The required statement now follows from the comparison theorem for series.
(d) Conclude that
\[
P\left( \bigcup_{n=1}^{\infty}\{X_k \leq k \text{ for all } k \geq n\}\right)
\]
= 1 in case (i) and = 0 in case (ii).
Solution: In case (i), this follows from
\[
P\left( \bigcup_{n=1}^{\infty}\{X_k \leq k \text{ for all } k \geq n\}\right) \geq P(X_k \leq k \text{ for all } k \geq m)
\]
for any \( m \). In case (ii), it follows from
\[
P\left( \bigcup_{n=1}^{\infty}\{X_k \leq k \text{ for all } k \geq n\}\right) \leq \sum_{n=1}^{\infty} P(X_k \leq k \text{ for all } k \geq n).
\]
Note that by applying this to the random variables \( X_n/\epsilon \), the case (i) statement can be strengthened to
\[
P\left( \bigcup_{n=1}^{\infty}\{X_k \leq \epsilon k \text{ for all } k \geq n\}\right) = 1
\]
By considering a sequence of \( \epsilon \)'s tending to 0, it can be further strengthened to
\[
P(\forall \epsilon > 0 \exists n \geq 1 \text{ such that } \forall k \geq n, X_k \leq \epsilon k) = 1.
\]
(e) Use part (d) to show that
\[
\frac{X_n}{n}
\]
converges to 0 a.s. in case (i) but not in case (ii).
Solution: In case (i), this now follows from part (d) and the definition of the limit: for every
\[
\omega \in \{\forall \epsilon > 0 \exists n \geq 1 \text{ such that } \forall k \geq n, X_k \leq \epsilon k\},
\]
\( X_n(\omega)/n \to 0 \). Case (ii) is similar.
Let $U$ be uniform on $[0, 1]$, and define random variables $X_1, X_2, \ldots$ by writing the decimal expansion of $U$ as

$$U = .X_1X_2X_3 \cdots.$$

(a) Show that $X_1, X_2, X_3$ are independent.

**Solution:** For $k = 0, 1, \ldots, 9$,

$$P(X_1 = k) = P\left(\frac{k}{10} < U < \frac{k+1}{10}\right) = \frac{1}{10}.$$ 

Similarly,

$$P(X_1 = k, X_2 = l, X_3 = m) = \frac{1}{10^3}.$$

(b) Let $P_n$ be the proportion of 3’s in the first $n$ decimal digits of $U$. Using the fact that the full sequence $X_1, X_2, \ldots$ is i.i.d., show that $P_n \to \frac{1}{10}$ a.s.

**Solution:** Let $Y_i$ be the indicator of the event $\{X_i = 3\}$. Then

$$P_n = \frac{1}{n} \sum_{i=1}^{n} Y_i.$$ 

Therefore, this follows from the SLLN.

(c) If we take the probability space to be $\Omega = [0, 1]$ with the usual assignment of probabilities and $U(\omega) = \omega$, is it true that $P_n \to \frac{1}{10}$ for every $\omega \in \Omega$? Explain.

**Solution:** No, e.g., $\omega = .5$.

(d) Let $Q_n$ be the proportion of 3’s in the first $n$ decimal digits of $U$ that are followed immediately by a 7. Show that $Q_n \to \frac{1}{100}$ a.s. (Suggestion: consider separately the even $k$’s for which $X_k = 3, X_{k+1} = 7$ and the odd $k$’s for which $X_k = 3, X_{k+1} = 7$.)

**Solution:** Now let $Y_i$ be the indicator of the event $\{X_i = 3, X_{i+1} = 7\}$. The $Y_i$’s are no longer independent, but the sequences $Y_1, Y_3, \ldots$ and $Y_2, Y_4, \ldots$ are each i.i.d. Therefore, by the SLLN,

$$\frac{1}{n} \sum_{i=0}^{n-1} Y_{2i+1} \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} Y_{2i}$$

each converges to $\frac{1}{100}$ a.s. It follows that

$$Q_{2n} = \frac{1}{2n} \sum_{i=0}^{n-1} Y_{2i+1} + \frac{1}{2n} \sum_{i=1}^{n} Y_{2i} \to \frac{1}{100} \quad a.s.$$ 

The same argument works for $Q_{2n+1}$.

You don’t need to show it, but the same argument can be used to show that for any finite block of digits (say 238 $\cdots$ 47), that block occurs
with limiting frequency $\frac{1}{10^n}$ a.s., where $n$ is the length of the block. A number in $[0, 1]$ is called normal to the base 10 if it has this property (for all finite blocks). An example of a normal number is obtained by listing the positive integers in order:

$.123456789101112131415161718 \cdots$

(e) Show that the set of normal numbers to the base 10 in $[0, 1]$ has probability 1. (This is known as the Borel Law of Normal Numbers.)

**Solution:** For each finite block $B$, let

$$A_B = \{\omega : B \text{ does not occur with the right limiting frequency in } \omega\}.$$ 

There are countably many such blocks, and $P(A_B) = 0$ for each $B$, so

$$P\left(\bigcup_B A_B\right) = 0.$$ 

Any $\omega \notin \bigcup_B A_B$ is normal.

Of course, the same is true for any base $b = 1, 2, 3, \ldots$. A number is called completely normal if it is normal to every base.

(f) Show that the set of completely normal numbers in $[0, 1]$ has probability 1.

**Solution:** The argument is the same as that for part (e), since there are countably many bases.

$K_2$. Consider a sequence of independent trials, each of which has three possible outcomes, $A, B, C$, with respective probabilities $p, q, r$ ($p + q + r = 1$). Find the probability of the event $D$ that an $A$ run of length $m$ occurs before a $B$ run of length $n$.

**Solution:** Let $u = P(D \mid X_1 = A)$, $v = P(D \mid X_1 = B)$, $w = P(D \mid X_1 = C) = P(D)$.

Then

$$u = \sum_{k=2}^{\infty} P(D \mid X_1 = A, \ldots, X_{k-1} = A, X_k = B)p^{k-2}q$$

$$+ \sum_{k=2}^{\infty} P(D \mid X_1 = A, \ldots, X_{k-1} = A, X_k = C)p^{k-2}r$$

$$= \sum_{k=2}^{m} vp^{k-2}q + \sum_{k=m+1}^{\infty} p^{k-2}q + \sum_{k=2}^{m} wp^{k-2}r + \sum_{k=m+1}^{\infty} p^{k-2}r$$

$$= qu + rw (1 - p^{m-1}) + p^{m-1}.$$
Similarly,
\[ v = \frac{pu + rw}{p + r} (1 - q^{n-1}). \]

Solving gives
\[ P(D) = w = \frac{(q + r)p^{m}(1 - q^{n})}{(q + r)p^{m} + (p + r)q^{n} - (p + q)p^{m}q^{n}}. \]

Recall that a Poisson process with parameter \( \lambda \) is a random collection of points on \([0, \infty)\) whose distribution is determined by the following equivalent properties:

(A) If \( T_1, T_2, \ldots \) are the successive spacings between points, then \( T_1, T_2, \ldots \) are i.i.d. with the exponential distribution with parameter \( \lambda \).

(B) If \( N(t) \) is the number of points in \([0, t]\), then for \( t_1 < t_2 < \cdots \), the random variables \( N(t_1), N(t_2) - N(t_1), N(t_3) - N(t_2), \ldots \) are independent Poisson random variables with parameters \( \lambda t_1, \lambda (t_2 - t_1), \lambda (t_3 - t_2), \ldots \).

In class, we checked part of the equivalence: (i) If (B) holds, then \( T_1 \) is Exponential \( (\lambda) \), and (ii) If (A) holds, then \( N(t) \) is Poisson \( (\lambda) \). In the next two problems, you will check another case of the equivalence.

K3. Suppose (B) holds.

(a) Write the event \( \{T_1 > s, T_1 + T_2 > s + t\} \) in terms of the random variables \( N(s) \) and \( N(s + t) \), and use this to compute its probability.

**Solution:**
\[ P(T_1 > s, T_1 + T_2 > s + t) = P(N(s) = 0, N(s + t) \leq 1) = e^{-\lambda(s+t)}(1+\lambda t). \]

(b) Write \( P(T_1 > s, T_1 + T_2 > s + t) \) in terms of the joint density of \( T_1 \) and \( T_2 \).

**Solution:** Letting \( f \) be the joint density,
\[ P(T_1 > s, T_1 + T_2 > s + t) = \int_{0}^{\infty} \int_{s}^{\infty} f(u, v) \, du \, dv - \int_{0}^{t} \int_{s}^{s+t-v} f(u, v) \, du \, dv. \]

(c) Use the fact that the answers to parts (a) and (b) are equal to show that \( T_1 \) and \( T_2 \) are independent Exponential \( (\lambda) \).

**Solution:** Equating the above expressions and differentiating with respect to \( s \) gives
\[ \lambda e^{-\lambda(s+t)}(1 + \lambda t) = \int_{0}^{t} f(s + t - v, v) \, dv + \int_{t}^{\infty} f(s, v) \, dv. \]
Differentiating this identity with respect to \( t \) gives (where \( f_1 \) is the partial derivative of \( f \) with respect to the first variable)
\[
\int_0^t f_1(w - v, v)dv = -\lambda^3 te^{-\lambda w},
\]
where \( w = s + t \). Differentiating with respect to \( t \) gives
\[
f_1(w - t, t) = -\lambda^3 e^{-\lambda w},
\]
i.e.
\[
f_1(s, t) = -\lambda^3 e^{-\lambda(s+t)}.
\]
Integrating gives
\[
f(s, t) = \lambda^2 e^{-\lambda(s+t)}.
\]

**K4.** Suppose (A) holds.

(a) Write the event \( \{N(s) = k, N(s+t) - N(s) = l\} \) in terms of the random variables \( T_1, T_2, \ldots \).

**Solution:** Letting \( S_n = T_1 + \cdots + T_n \),
\[
P(N(s) = k, N(s+t) - N(s) = l) = P(S_k < s < S_{k+1}, S_{k+l} < s+t < S_{k+l+1}).
\]

(b) Use the fact that the sum of \( k \) independent Exponential (\( \lambda \)) distributed random variables is Gamma (\( k, \lambda \)) to show that \( N(s) \) and \( N(s+t) - N(s) \)

are independent Poisson distributed random variables with parameters \( \lambda s \) and \( \lambda t \) respectively.

**Solution:** Conditioning on the values of \( S_k, T_{k+1}, S_{k+l} - S_{k+1} \), and letting \( f_k(x) \) be the Gamma\((k, \lambda)\) density, gives the following expression for the above probability: (WLOG, assume \( l \geq 1 \))
\[
\int \int \int_A f_k(x)f_1(y)f_{l-1}(z)e^{-\lambda(s+t-x-y-z)}dzdydx,
\]
where \( A = \{x < s < x + y, x + y + z < s + t\} \). The integrand is
\[
e^{-\lambda(s+\theta)} \frac{\lambda^{k+l}}{(k-1)!(l-1)!}e^{k-1}z^{l-2}.
\]
Integrating on \( 0 < z < s + t - x - y \) gives the following expression for the integral:
\[
e^{-\lambda(s+t)} \frac{\lambda^{k+l}}{(k-1)!(l-1)!} \int_{x<s<x+y<s+t} x^{k-1}(s + t - y)^{l-1}dydx.
\]
Integrating \( s - x < y < s + t - x \) gives
\[
e^{-\lambda(s+t)} \frac{\lambda^{k+l}}{(k-1)!(l-1)!} \int_0^s x^{k-1}t^{l-1} \frac{dx}{t}.
\]
Integrating on $0 < x < s$ gives

$$e^{-\lambda (s+t)} \frac{\lambda^{k+l}}{k!l!} t^l s^k$$

as required.