Mathematics 170B – Selected HW Solutions.

F_4. Suppose X_n is $B(n,p)$.
(a) Find the moment generating function $M_n(s)$ of $(X_n - np)/\sqrt{np(1-p)}$.

Write $q = 1-p$. The MGF of X_n is $(pe^s+q)^n$, since X_n can be written as the sum of n independent Bernoulli's with parameter p, and these have MGF pe^s+q. Therefore,

$$M_n(s) = E\exp\left\{s\frac{X_n-np}{\sqrt{npq}}\right\} = e^{-s\sqrt{np/q}}\big[pe^{s/\sqrt{npq}}+q\big]^n = \big[pe^{s\sqrt{q/np}}+qe^{-s\sqrt{p/nq}}\big]^n.$$

(b) Compute the limit

$$\lim_{n\to\infty} M_n(s),$$

directly, without using the central limit theorem.

We want to write $M_n(s)$ in the form $(1+\frac{a_n}{n})^n$. Solving for a_n gives

$$a_n = n\big[pe^{s\sqrt{q/np}} + qe^{-s\sqrt{p/nq}} - 1\big].$$

Recalling that the expansion of the exponential is $e^x = 1+x+x^2/2+\cdots$ suggests that this should be rewritten in the form

$$a_n = np\big[e^{s\sqrt{q/np}} - 1 - s\sqrt{q/np}\big] + nq\big[e^{-s\sqrt{p/nq}} - 1 + s\sqrt{p/nq}\big].$$

Since

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2} = \frac{1}{2} \qquad (1)$$

(by applying L'Hopital's rule twice),

$$\lim_{n\to\infty} a_n = \frac{s^2}{2}.$$

So,

$$\lim_{n\to\infty} M_n(s) = e^{s^2/2},$$

which is the mgf of the $N(0,1)$ distribution.

F_5. Suppose X_n is Poisson with parameter n.
(a) Find the moment generating function $M_n(s)$ of $(X_n-n)/\sqrt{n}$.

Since the MGF of the Poisson with parameter n is

$$e^{n(e^s-1)},$$

$$M_n(s) = e^{-s\sqrt{n}+n(e^{s/\sqrt{n}}-1)}.$$

(b) Compute the limit

$$\lim_{n\to\infty} M_n(s),$$

directly, without using the central limit theorem.

Taking logs gives

$$\log M_n(s) = n\left[e^{s/\sqrt{n}} - \frac{s}{\sqrt{n}} - 1\right],$$

so using (1) again gives

$$\lim_{n\to\infty} \log M_n(s) = \frac{s^2}{2},$$

so that

$$\lim_{n\to\infty} M_n(s) = e^{s^2/2}$$

as in Problem F_4.

G_2. Suppose the random variables X_n satisfy $EX_n = 0$, $EX_n^2 \leq 1$, and $Cov(X_n, X_m) \leq 0$ for $n \neq m$. Show that

$$\frac{S_n}{n} = \frac{X_1 + \cdots + X_n}{n}$$

converges to 0 in probability.

Solution: The proof follows the proof of the WLLN under a second moment assumption: By Chebyshev,

$$P(|S_n/n| \geq \epsilon) \leq \frac{1}{\epsilon^2 n^2} var(S_n).$$

But

$$var(S_n) = \sum_{i,j=1}^{n} cov(X_i, X_j) \leq \sum_{i=1}^{n} var(X_i) \leq n.$$

Combining these gives

$$P(|S_n/n| \geq \epsilon) \leq \frac{1}{\epsilon^2 n},$$

which tends to zero as $n \to \infty$.

G_3. Show that in each of the cases (a), (c), and (d) of Problem 5 on page 288, the sequence actually converges a.s.

Solution: For (a): $EY_n^2 = \frac{1}{3n^2}$, so $\sum_n EY_n^2 < \infty$. Therefore $Y_n^2 \to 0$ a.s., so $Y_n \to 0$ a.s. For (c), $E|Y_n| = \frac{1}{2^n}$, so $\sum_n E|Y_n| < \infty$. Therefore $Y_n \to 0$ a.s. For (d), there are two possible approaches: One is to show that $E(1 - Y_n)^2 = \frac{8}{(n+1)(n+2)}$, and proceed as in the other cases. The other is to note that Y_n is nondecreasing in n and is bounded above by 1. Therefore, $Y = \lim_{n\to\infty} Y_n$ exists for every ω, and satisfies $Y \leq 1$. To show that $Y = 1$ a.s., take $0 < \epsilon < 1$ and write

$$P(Y \leq 1 - \epsilon) \leq P(Y_n \leq 1 - \epsilon) = (1 - (\epsilon/2))^n,$$

which tends to zero as $n \to \infty$. Therefore $P(Y < 1) = \lim_{\epsilon\downarrow 0} P(Y \leq 1 - \epsilon) = 0$.

G_4. Suppose each X_n takes the values ± 1 with probability $\frac{1}{2}$ each. Show that the random series

$$\sum_{n=1}^{\infty} \frac{X_n}{n^p}$$

converges a.s. (which means that the partial sums converge a.s.) if $p > 1$.

Solution: The series converges absolutely, since

$$\sum_{n=1}^{\infty} \frac{|X_n|}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty.$$

Now use the fact that absolute convergence of a series implies convergence.

H_1. Suppose X_n are i.i.d. non-negative random variables.

(a) Show that

$$\frac{X_n}{n} \to 0$$

in probability with no further assumptions. (You did this before in case they are uniformly distributed on $[-1, 1]$.)

Solution:

$$P(X_n/n > \epsilon) = P(X_1 > n\epsilon) \to 0$$

as $n \to \infty$.

Consider now two cases: (i) $EX < \infty$ and (ii) $EX = \infty$. Recall that the series

$$\sum_k P(X_1 > k) = \sum_k P(X_k > k)$$

converges in case (i) and diverges in case (ii). (See Problem 3 on page 184. This gives the statement in terms of integrals rather than sums, but there is no real difference.)

(b) Express

$$P(X_k \leq k \text{ for all } k \geq n)$$

in terms of the probabilities $P(X_k > k)$.

Solution:

$$\prod_{k=n}^{\infty} [1 - P(X_1 > k)].$$

(c) Show that

$$\lim_{n \to \infty} P(X_k \leq k \text{ for all } k \geq n) = 1$$

in case (i) and $P(X_k \leq k \text{ for all } k \geq n) = 0$ for all n in case (ii). (Suggestion: take logs.)

Solution: Since $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$ by L'Hopital, there is an $\epsilon > 0$ so that

$$\frac{x}{2} \le |\log(1-x)| \le \frac{3x}{2} \quad \text{for } 0 \le x \le \epsilon.$$

Therefore, for sufficiently large k,

$$\frac{1}{2}P(X_1 > k) \le |\log[1 - P(X_1 > k)]| \le \frac{3}{2}P(X_1 > k).$$

The required statement now follows from the comparison theorem for series.

(d) Conclude that

$$P\left(\bigcup_{n=1}^{\infty}\{X_k \le k \text{ for all } k \ge n\}\right)$$

$= 1$ in case (i) and $= 0$ in case (ii).
Solution: In case (i), this follows from

$$P\left(\bigcup_{n=1}^{\infty}\{X_k \le k \text{ for all } k \ge n\}\right) \ge P(X_k \le k \text{ for all } k \ge m)$$

for any m. In case (ii), it follows from

$$P\left(\bigcup_{n=1}^{\infty}\{X_k \le k \text{ for all } k \ge n\}\right) \le \sum_{n=1}^{\infty} P(X_k \le k \text{ for all } k \ge n).$$

Note that by applying this to the random variables X_n/ϵ, the case (i) statement can be strengthened to

$$P\left(\bigcup_{n=1}^{\infty}\{X_k \le \epsilon k \text{ for all } k \ge n\}\right) = 1$$

By considering a sequence of ϵ's tending to 0, it can be further strengthened to

$$P(\forall \epsilon > 0 \ \exists n \ge 1 \text{ such that } \ \forall k \ge n, X_k \le \epsilon k) = 1.$$

(e) Use part (d) to show that

$$\frac{X_n}{n}$$

converges to 0 a.s. in case (i) but not in case (ii).
Solution: In case (i), this now follows from part (d) and the definition of the limit: for every

$$\omega \in \{\forall \epsilon > 0 \ \exists n \ge 1 \text{ such that } \ \forall k \ge n, X_k \le \epsilon k\},$$

$X_n(\omega)/n \to 0$. Case (ii) is similar.

H_2. Let U be uniform on $[0,1]$, and define random variables $X_1, X_2, ...$ by writing the decimal expansion of U as

$$U = .X_1 X_2 X_3 \cdots .$$

(a) Show that X_1, X_2, X_3 are independent.
Solution: For $k = 0, 1, \ldots, 9$,

$$P(X_1 = k) = P\left(\frac{k}{10} < U < \frac{k+1}{10}\right) = \frac{1}{10}.$$

Similarly,

$$P(X_1 = k, X_2 = l, X_3 = m) = \frac{1}{10^3}.$$

(b) Let P_n be the proportion of 3's in the first n decimal digits of U. Using the fact that the full sequence $X_1, X_2, \ldots$ is i.i.d., show that $P_n \to \frac{1}{10}$ a.s.
Solution: Let Y_i be the indicator of the event $\{X_i = 3\}$. Then

$$P_n = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

Therefore, this follows from the SLLN.

(c) If we take the probability space to be $\Omega = [0,1]$ with the usual assignment of probabilities and $U(\omega) = \omega$, is it true that $P_n \to \frac{1}{10}$ for **every** $\omega \in \Omega$? Explain.
Solution: No, e.g., $\omega = .5$.

(d) Let Q_n be the proportion of 3's in the first n decimal digits of U that are followed immediately by a 7. Show that $Q_n \to \frac{1}{100}$ a.s. (Suggestion: consider separately the even k's for which $X_k = 3, X_{k+1} = 7$ and the odd k's for which $X_k = 3, X_{k+1} = 7$.)
Solution: Now let Y_i be the indicator of the event $\{X_i = 3, X_{i+1} = 7\}$. The Y_i's are no longer independent, but the sequences $Y_1, Y_3, \ldots$ and $Y_2, Y_4, \ldots$ are each i.i.d. Therefore, by the SLLN,

$$\frac{1}{n}\sum_{i=0}^{n-1} Y_{2i+1} \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n} Y_{2i}$$

each converges to $\frac{1}{100}$ a.s. It follows that

$$Q_{2n} = \frac{1}{2n}\sum_{i=0}^{n-1} Y_{2i+1} + \frac{1}{2n}\sum_{i=1}^{n} Y_{2i} \to \frac{1}{100} \quad a.s.$$

The same argument works for Q_{2n+1}.

You don't need to show it, but the same argument can be used to show that for any finite block of digits (say $238 \cdots 47$), that block occurs with limiting frequency $\frac{1}{10^n}$ a.s., where n is the length of the block. A number in $[0,1]$ is called normal to the base 10 if it has this property (for all finite blocks). An example of a normal number is obtained by listing the positive integers in order:

$$.123456789101112131415161718\cdots.$$

(e) Show that the set of normal numbers to the base 10 in [0,1] has probability 1. (This is known as the Borel Law of Normal Numbers.)
Solution: For each finite block B, let

$$A_B = \{\omega : B \text{ does not occur with the right limiting frequency in } \omega\}.$$

There are countably many such blocks, and $P(A_B) = 0$ for each B, so

$$P\left(\bigcup_B A_B\right) = 0.$$

Any $\omega \notin \cup_B A_B$ is normal.

Of course, the same is true for any base $b = 1, 2, 3, \ldots$. A number is called completely normal if it is normal to every base.

(f) Show that the set of completely normal numbers in [0,1] has probability 1.
Solution: The argument is the same as that for part (e), since there are countably many bases.

K_2. Consider a sequence of independent trials, each of which has three possible outcomes, A, B, C, with respective probabilities p, q, r $(p + q + r = 1)$. Find the probability of the event D that an A run of length m occurs before a B run of length n.

Solution: Let

$$u = P(D \mid X_1 = A),\ v = P(D \mid X_1 = B),\ w = P(D \mid X_1 = C) = P(D).$$

Then

$$\begin{aligned}
u &= \sum_{k=2}^{\infty} P(D \mid X_1 = A, \ldots, X_{k-1} = A, X_k = B)p^{k-2}q \\
&\quad + \sum_{k=2}^{\infty} P(D \mid X_1 = A, \ldots, X_{k-1} = A, X_k = C)p^{k-2}r \\
&= \sum_{k=2}^{m} vp^{k-2}q + \sum_{k=m+1}^{\infty} p^{k-2}q + \sum_{k=2}^{m} wp^{k-2}r + \sum_{k=m+1}^{\infty} p^{k-2}r \\
&= \frac{qv + rw}{q + r}(1 - p^{m-1}) + p^{m-1}.
\end{aligned}$$

Similarly,

$$v = \frac{pu + rw}{p + r}(1 - q^{n-1}).$$

Solving gives

$$P(D) = w = \frac{(q + r)p^m(1 - q^n)}{(q + r)p^m + (p + r)q^n - (p + q)p^m q^n}.$$

Recall that a Poisson process with parameter λ is a random collection of points on $[0, \infty)$ whose distribution is determined by the following equivalent properties:

(A) If $T_1, T_2, \ldots$ are the successive spacings between points, then $T_1, T_2, \ldots$ are i.i.d. with the exponential distribution with parameter λ.

(B) If $N(t)$ is the number of points in $[0, t]$, then for $t_1 < t_2 < \cdots$, the random variables $N(t_1), N(t_2) - N(t_1), N(t_3) - N(t_2), \ldots$ are independent Poisson random variables with parameters $\lambda t_1, \lambda(t_2 - t_1), \lambda(t_3 - t_2), \ldots$.

In class, we checked part of the equivalence: (i) If (B) holds, then T_1 is Exponential (λ), and (ii) If (A) holds, then $N(t)$ is Poisson (λ). In the next two problems, you will check another case of the equivalence.

K_3. Suppose (B) holds.

(a) Write the event $\{T_1 > s, T_1 + T_2 > s + t\}$ in terms of the random variables $N(s)$ and $N(s + t)$, and use this to compute its probability.

Solution:

$$P(T_1 > s, T_1 + T_2 > s + t) = P(N(s) = 0, N(s + t) \leq 1) = e^{-\lambda(s+t)}(1 + \lambda t).$$

(b) Write $P(T_1 > s, T_1 + T_2 > s + t)$ in terms of the joint density of T_1 and T_2.

Solution: Letting f be the joint density,

$$P(T_1 > s, T_1 + T_2 > s + t) = \int_0^\infty \int_s^\infty f(u, v)\,du\,dv - \int_0^t \int_s^{s+t-v} f(u, v)\,du\,dv.$$

(c) Use the fact that the answers to parts (a) and (b) are equal to show that T_1 and T_2 are independent Exponential (λ).

Solution: Equating the above expressions and differentiating with respect to s gives

$$\lambda e^{-\lambda(s+t)}(1 + \lambda t) = \int_0^t f(s + t - v, v)\,dv + \int_t^\infty f(s, v)\,dv.$$

Differentiating this identity with respect to t gives (where f_1 is the partial derivative of f with respect to the first variable)

$$\int_0^t f_1(w-v,v)dv = -\lambda^3 te^{-\lambda w},$$

where $w = s+t$. Differentiating with respect to t gives

$$f_1(w-t,t) = -\lambda^3 e^{-\lambda w},$$

i.e.

$$f_1(s,t) = -\lambda^3 e^{-\lambda(s+t)}.$$

Integrating gives

$$f(s,t) = \lambda^2 e^{-\lambda(s+t)}.$$

K_4. Suppose (A) holds.

(a) Write the event $\{N(s) = k, N(s+t) - N(s) = l\}$ in terms of the random variables $T_1, T_2, \ldots$.

Solution: Letting $S_n = T_1 + \cdots + T_n$,

$$P(N(s) = k, N(s+t) - N(s) = l) = P(S_k < s < S_{k+1}, S_{k+l} < s+t < S_{k+l+1}).$$

(b) Use the fact that the sum of k independent Exponential (λ) distributed random variables is Gamma (k,λ) to show that $N(s)$ and

$$N(s+t) - N(s)$$

are independent Poisson distributed random variables with parameters λs and λt respectively.

Solution: Conditioning on the values of $S_k, T_{k+1}, S_{k+l} - S_{k+1}$, and letting $f_k(x)$ be the Gamma(k,λ) density, gives the following expression for the above probability: (WLOG, assume $l \geq 1$)

$$\int\int\int_A f_k(x)f_1(y)f_{l-1}(z)e^{-\lambda(s+t-x-y-z)}dzdydx,$$

where $A = \{x < s < x+y, x+y+z < s+t\}$. The integrand is

$$e^{-\lambda(s+t)}\frac{\lambda^{k+l}}{(k-1)!(l-2)!}x^{k-1}z^{l-2}.$$

Integrating on $0 < z < s+t-x-y$ gives the following expression for the integral:

$$e^{-\lambda(s+t)}\frac{\lambda^{k+l}}{(k-1)!(l-1)!}\int\int_{x<s<x+y<s+t} x^{k-1}(s+t-x-y)^{l-1}dydx.$$

Integrating $s-x < y < s+t-x$ gives

$$e^{-\lambda(s+t)}\frac{\lambda^{k+l}}{(k-1)!(l-1)!}\int_0^s x^{k-1}\frac{t^l}{l}dx.$$

Integrating on $0 < x < s$ gives

$$e^{-\lambda(s+t)}\frac{\lambda^{k+l}}{k!l!}t^l s^k$$

as required.