
Problems 1,2,3 on page 326.

Definition. A positive integer valued random variable \( N \) is a stopping time with respect to the discrete time stochastic process \( \{X_1, X_2, \ldots \} \) if for every \( n \), the event \( \{N = n\} \) is determined by the values of \( \{X_1, X_2, \ldots, X_n\} \).

\( J_1 \). Suppose \( \{X_1, X_2, \ldots \} \) is a Bernoulli process. Show that \( N \), the time at which the second success occurs is a stopping time.

\( J_2 \). Show that if \( N \) is a stopping time, then so is \( N_k = \min\{N, k\} \).

Wald’s second identity states the following: If \( X_1, X_2, \ldots \) are i.i.d. and satisfy \( EX_i = 0, EX_i^2 = \sigma^2 < \infty \), and \( N \) is a stopping time with \( EN < \infty \), then \( ES_N^2 = \sigma^2 EN \). You need not prove this; those of you in Math 171 may have done the proof.

\( J_3 \). Consider the simple symmetric random walk \( S_n = X_1 + \cdots + X_n \), where \( X_1, X_2, \ldots \) are i.i.d. random variables taking the values \( \pm 1 \) with probability \( \frac{1}{2} \) each. For positive integers \( l, m \), let

\[
N = \min\{n \geq 1 : S_n = -l \text{ or } m\}
\]

and \( N_k = \min\{N, k\} \). (\( N \), and hence \( N_k \), is a stopping time, but you need not prove this – it should be obvious to you!)

(a) Use Wald’s second identity to obtain an upper bound for \( EN_k \) that does not depend on \( k \).
(b) Conclude that \( EN < \infty \).
(c) Apply Wald’s first identity to compute \( P(S_N = m) \).
(d) Use Wald’s second identity to compute \( EN \).