H1. Suppose $X_n$ are i.i.d. non-negative random variables.
(a) Show that
\[
\frac{X_n}{n} \rightarrow 0
\]
in probability with no further assumptions. (You did this before in case they are uniformly distributed on $[-1, 1]$.)

Consider now two cases: (i) $EX < \infty$ and (ii) $EX = \infty$. Recall that the series
\[
\sum_k P(X_1 > k) = \sum_k P(X_k > k)
\]
converges in case (i) and diverges in case (ii). (See Problem 3 on page 184. This gives the statement in terms of integrals rather than sums, but there is no real difference.)
(b) Express
\[
P(X_k \leq k \text{ for all } k \geq n)
\]
in terms of the probabilities $P(X_k > k)$.
(c) Show that
\[
\lim_{n \rightarrow \infty} P(X_k \leq k \text{ for all } k \geq n) = 1
\]
in case (i) and $P(X_k \leq k \text{ for all } k \geq n) = 0$ for all $n$ in case (ii). (Suggestion: take logs.)
(d) Conclude that
\[
P\left(\bigcup_{n=1}^{\infty}\{X_k \leq k \text{ for all } k \geq n\}\right)
\]
= 1 in case (i) and = 0 in case (ii). Note that by applying this to the random variables $X_n/\epsilon$, the case (i) statement can be strengthened to
\[
P\left(\bigcup_{n=1}^{\infty}\{X_k \leq \epsilon k \text{ for all } k \geq n\}\right) = 1
\]
(e) Use part (d) to show that
\[
\frac{X_n}{n}
\]
converges to 0 a.s. in case (i) but not in case (ii).

H2. Let $U$ be uniform on $[0, 1]$, and define random variables $X_1, X_2, \ldots$ by writing the decimal expansion of $U$ as
(a) Show that $X_1, X_2, X_3$ are independent.

(b) Let $P_n$ be the proportion of 3’s in the first $n$ decimal digits of $U$. Using the fact that the full sequence $X_1, X_2, \ldots$ is i.i.d., show that $P_n \to \frac{1}{10}$ a.s.

(c) If we take the probability space to be $\Omega = [0, 1]$ with the usual assignment of probabilities and $U(\omega) = \omega$, is it true that $P_n \to \frac{1}{10}$ for every $\omega \in \Omega$? Explain.

(d) Let $Q_n$ be the proportion of 3’s in the first $n$ decimal digits of $U$ that are followed immediately by a 7. Show that $Q_n \to \frac{1}{100}$ a.s. (Suggestion: consider separately the even $k$’s for which $X_k = 3, X_{k+1} = 7$ and the odd $k$’s for which $X_k = 3, X_{k+1} = 7$.)

You don’t need to show it, but the same argument can be used to show that for any finite block of digits (say 238· · ·47), that block occurs with limiting frequency $\frac{1}{10^n}$ a.s., where $n$ is the length of the block. A number in $[0, 1]$ is called normal to the base 10 if it has this property (for all finite blocks). An example of a normal number is obtained by listing the positive integers in order:

\[
.123456789101112131415161718\ldots
\]

(e) Show that the set of normal numbers to the base 10 in $[0,1]$ has probability 1. (This is known as the Borel Law of Normal Numbers.)

Of course, the same is true for any base $b = 1, 2, 3, \ldots$. A number is called completely normal if it is normal to every base.

(f) Show that the set of completely normal numbers in $[0,1]$ has probability 1.

Note: There is no known example of a completely normal number, even though if you choose a number in $[0,1]$ “at random”, it will be completely normal with probability 1.