Mathematics 170B – HW8 – Due Tuesday, May 22, 2012.

H_1. Suppose X_n are i.i.d. non-negative random variables.

(a) Show that

$$\frac{X_n}{n} \to 0$$

in probability with no further assumptions. (You did this before in case they are uniformly distributed on $[-1, 1]$.)

Consider now two cases: (i) $EX < \infty$ and (ii) $EX = \infty$. Recall that the series

$$\sum_k P(X_1 > k) = \sum_k P(X_k > k)$$

converges in case (i) and diverges in case (ii). (See Problem 3 on page 184. This gives the statement in terms of integrals rather than sums, but there is no real difference.)

(b) Express

$$P(X_k \leq k \text{ for all } k \geq n)$$

in terms of the probabilities $P(X_k > k)$.

(c) Show that

$$\lim_{n\to\infty} P(X_k \leq k \text{ for all } k \geq n) = 1$$

in case (i) and $P(X_k \leq k \text{ for all } k \geq n) = 0$ for all n in case (ii). (Suggestion: take logs.)

(d) Conclude that

$$P\left(\bigcup_{n=1}^{\infty}\{X_k \leq k \text{ for all } k \geq n\}\right)$$

$= 1$ in case (i) and $= 0$ in case (ii). Note that by applying this to the random variables X_n/ϵ, the case (i) statement can be strengthened to

$$P\left(\bigcup_{n=1}^{\infty}\{X_k \leq \epsilon k \text{ for all } k \geq n\}\right) = 1$$

(e) Use part (d) to show that

$$\frac{X_n}{n}$$

converges to 0 a.s. in case (i) but not in case (ii).

H_2. Let U be uniform on $[0, 1]$, and define random variables $X_1, X_2, \ldots$ by writing the decimal expansion of U as

$$U = .X_1X_2X_3\cdots.$$

(a) Show that X_1, X_2, X_3 are independent.

(b) Let P_n be the proportion of 3's in the first n decimal digits of U. Using the fact that the full sequence $X_1, X_2, \ldots$ is i.i.d., show that $P_n \to \frac{1}{10}$ a.s.

(c) If we take the probability space to be $\Omega = [0,1]$ with the usual assignment of probabilities and $U(\omega) = \omega$, is it true that $P_n \to \frac{1}{10}$ for **every** $\omega \in \Omega$? Explain.

(d) Let Q_n be the proportion of 3's in the first n decimal digits of U that are followed immediately by a 7. Show that $Q_n \to \frac{1}{100}$ a.s. (Suggestion: consider separately the even k's for which $X_k = 3, X_{k+1} = 7$ and the odd k's for which $X_k = 3, X_{k+1} = 7$.)

You don't need to show it, but the same argument can be used to show that for any finite block of digits (say $238\cdots47$), that block occurs with limiting frequency $\frac{1}{10^n}$ a.s., where n is the length of the block. A number in $[0,1]$ is called normal to the base 10 if it has this property (for all finite blocks). An example of a normal number is obtained by listing the positive integers in order:

$$.123456789101112131415161718\cdots.$$

(e) Show that the set of normal numbers to the base 10 in [0,1] has probability 1. (This is known as the Borel Law of Normal Numbers.)

Of course, the same is true for any base $b = 1, 2, 3, \ldots$. A number is called completely normal if it is normal to every base.

(f) Show that the set of completely normal numbers in [0,1] has probability 1.

Note: There is no known example of a completely normal number, even though if you choose a number in [0,1] "at random", it will be completely normal with probability 1.