Mathematics 170B – HW5 – Due Tuesday, May 1, 2012.

Problems 41 and 42 on page 260 and problem 1 on page 284.

E_1. Show that for any random variable X,

$$P(|X| \geq a) \leq \frac{EX^4}{a^4}, \quad a > 0$$

in two different ways:
(a) Deduce it from Markov's inequality.
(b) Prove it directly in the same way Markov's inequality is proved.

E_2. Suppose X is Poisson with parameter λ. Use Chebyshev's inequality to show:
(a) $P(X \leq \lambda/2) \leq 4/\lambda$.
(b) $P(X \geq 2\lambda) \leq 1/\lambda$.

E_3. Suppose X is Poisson with parameter λ.
(a) Apply the result of Problem 2(a) on page 284 to get an upper bound for $P(X \geq 2\lambda)$.
(b) The bound in part (a) above depends on s. Choose the s that makes this bound as small as possible to show that $P(X \geq 2\lambda) \leq (e/4)^\lambda$.
(c) Compare the bounds obtained in problems E_2(b) and E_3(b) when $\lambda = 10$.

E_4. Suppose $X_1, X_2, ...$ are i.i.d. random variables with $P(X_i = 1) = p, P(X_i = -1) = 1 - p$, and let $S_n = X_1 + \cdots + X_n$ be their partial sums. This is called a simple random walk. For $k = 1, 2, ...$, let $N_k = \min\{n \geq 1 : S_n = -k\}$ be the first time that the random walk hits $-k$, and let $M_k(s)$ be the moment generating function of N_k. Note that the random variables $N_1, N_2 - N_1, N_3 - N_2, \ldots$ are i.i.d.
(a) Express $M_k(s)$ in terms of $M_1(s)$.
(b) By conditioning on the value of X_1, find an equation relating $M_1(s)$ and $M_2(s)$.
(c) Combine the results of (a) and (b) to find an equation satisfied by $M_1(s)$.
(d) Assuming $M_1(s) < \infty$, solve the equation from part (c) for $M_1(s)$. (To resolve the sign ambiguity, use the fact that $\lim_{s \to -\infty} M(s) = 0$.)
(e) Use the result of part (d) to compute EN_1 for $p < \frac{1}{2}$. What do you think happens to this if $p = \frac{1}{2}$?