Mathematics 170B – HW4 – Due Tuesday April 24, 2012.

Problems 29, 30, 32, 33, 35 on pages 256-257.

D_1. For a group of n people, let X_k be the number of days of the year (assuming 365 days) which are the birthdays of exactly k people $(0 \leq k \leq n)$.
(a) Find EX_k.
(b) Find $\text{var}(X_k)$.
(c) Find $\text{cov}(X_k, X_l)$ for $k \neq l$.

D_2. Let $X_1, X_2, ..., X_n$ be i.i.d. random variables with mean μ and variance σ^2. Let

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n} \quad \text{and} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

be the sample mean and sample variance respectively. Show that $ES^2 = \sigma^2$. (In statistical language, this says that S^2 is an unbiased estimator of σ^2. Note that for this to be the case, the denominator in S^2 must be $n-1$ rather than n.)

D_3. In class, we considered the following problem: Let $U_1, U_2, ...$ be i.i.d. random variables with the uniform distribution on $[0, 1]$. For $x > 0$, define

$$N(x) = \min\left\{n : \sum_{i=1}^{n} U_i > x\right\} \quad \text{and} \quad m(x) = EN(x).$$

By using the law of iterated expectations, we derived the relation

$$m(x) = 1 + \int_0^1 m(x-u)du,$$

and concluded that $m(x) = e^x$ for $0 < x \leq 1$. Compute $m(x)$ for $1 \leq x \leq 2$.

D_4. Recall that the Gamma density with parameters $\alpha > 0$ and $\lambda > 0$ is given by

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0.$$

Find its moment generating function $M(s)$. For what values of s is $M(s) < \infty$?

D_5. In class, we used convolutions to show that the sum of two independent Gamma distributed random variables with parameters α, λ and β, λ respectively is Gamma with parameters $\alpha + \beta, \lambda$. Show this using problem D_4 above.