Mathematics 170B – HW10 – Due Tuesday, June 5, 2012.

Problems # 8, 9,11,12 on pages 329-30.

Definition. In a Bernoulli process, a success run of length k is a sequence of the form $\cdots 11111 \cdots$, where there are k consecutive 1's. Similarly, a failure run of length k is a sequence of the form $\cdots 00000 \cdots$, where there are k consecutive 0's.

In class, we will see that the probability of having a success run of length m before a failure run of length n is given by

$$p^{m-1} \frac{1 - q^n}{p^{m-1} + q^{n-1} - p^{m-1}q^{n-1}},$$

where $q = 1 - p$.

K_1. Find an integer k so that in successive rolls of a fair die, the probability is about $\frac{1}{2}$ that a run of three consecutive 6's appears before a run of k consecutive non 6's.

K_2. Consider a sequence of independent trials, each of which has three possible outcomes, A, B, C, with respective probabilities p, q, r ($p+q+r = 1$). Find the probability that an A run of length m occurs before a B run of length n.

Recall that a Poisson process with parameter λ is a random collection of points on $[0, \infty)$ whose distribution is determined by the following equivalent properties:

(A) If $T_1, T_2, \ldots$ are the successive spacings between points, then $T_1, T_2, \ldots$ are i.i.d. with the exponential distribution with parameter λ.

(B) If $N(t)$ is the number of points in $[0, t]$, then for $t_1 < t_2 < \cdots$, the random variables $N(t_1), N(t_2) - N(t_1), N(t_3) - N(t_2), \ldots$ are independent Poisson random variables with parameters $\lambda t_1, \lambda(t_2 - t_1), \lambda(t_3 - t_2), \ldots$.

In class, we checked part of the equivalence: (i) If (B) holds, then T_1 is Exponential (λ), and (ii) If (A) holds, then $N(t)$ is Poisson (λ). In the next two problems, you will check another case of the equivalence.

K_3. Suppose (B) holds.

(a) Write the event $\{T_1 > s, T_1 + T_2 > s + t\}$ in terms of the random variables $N(s)$ and $N(s + t)$, and use this to compute its probability.

(b) Write $P(T_1 > s, T_1 + T_2 > s + t)$ in terms of the joint density of T_1 and T_2.

(c) Use the fact that the answers to parts (a) and (b) are equal to show that T_1 and T_2 are independent Exponential (λ).

K_4. Suppose (A) holds.

(a) Write the event $\{N(s) = k, N(s+t) - N(s) = l\}$ in terms of the random variables $T_1, T_2, \ldots$.

(b) Use the fact that the sum of k independent Exponential (λ) distributed random variables is Gamma (k, λ) to show that $N(s)$ and

$$N(s+t) - N(s)$$

are independent Poisson distributed random variables with parameters λs and λt respectively.

K_5. In class we showed that the conditional distribution of T_1 given $\{N(t) = 1\}$ is $U[0, t]$. Show that the conditional distribution of $(T_1, T_1 + T_2)$ given $\{N(t) = 2\}$ is the same of the order statistics of two independent $U[0, t]$ distributed random variables.