T. Liggett Mathematics 170B – Final Exam Solutions June 13, 2012

(16) 1. Let $N(t)$ be a Poisson process with rate $\lambda = 2$, and for $0 \le a < b$, let $N(a, b) = N(b) - N(a)$ be the number of Poisson points in the interval (a, b).

(a) Find $P(N(2, 3) = 6 \mid N(0, 5) = 10)$.

Solution: $\binom{10}{6}(1/5)^6(4/5)^4$.

(b) Find $Cov(N(0, 4), N(3, 5))$.

Solution: Write $N(0, 4) = N(0, 3)+N(3, 4)$ and $N(3, 5) = N(3, 4)+N(4, 5)$. By independence over disjoint sets and bilinearity of covariance, we have $Cov(N(0, 4), N(3, 5)) = Var(N(3, 4)) = 2$.

(16) 2. Decide whether each statement is true or false. No explanation is needed. Scoring: +2 for each correct answer, −1 for each incorrect answer, 0 for no answer.

(a) If X,Y are independent Poisson distributed random variables then $X + Y$ is Poisson distributed.

Solution: True.

(b) If X,Y are independent Poisson distributed random variables then $X - Y$ is Poisson distributed.

Solution: False.

(c) If $N(t)$ has a Poisson distribution with parameter t whenever $t \ge 0$, then $N(t)$ is a Poisson process.

Solution: False.

(d) If X_n converges to X in distribution, then X_n converges to X in probability.

Solution: False.

(e) If $EX^2 < \infty$, then $EX^4 < \infty$.

Solution: False.

(f) Suppose $X_1, X_2, \dots$ are i.i.d. with $P(X_i = -1) = P(X_i = +1) = \frac{1}{2}$, and let $S_n = X_1 + \cdots + X_n$, and $N = \min\{n \ge 1 : S_n = 0\}$. Then $EN < \infty$.

Solution: False.

(g) In the context of (f), $P(S_n = -10^7 \text{ for some } n) = 1$.

Solution: True.

(h) If $M(s)$ is the MGF of a random variable X, then there is an $a > 0$ so that $M(s) < \infty$ for all $-a < s < a$.

Solution: False.

(20) 3. A fair die is tossed n times. Let X be the number of 1's and 2's obtained, and Y be the number of 6's obtained. Find the covariance and correlation coefficient of X and Y without using the joint distribution of X and Y.

Solution: Write $X = X_1 + \cdots + X_n$ and $Y = Y_1 + \cdots + Y_n$ where X_i and Y_i are the indicators of {get 1 or 2 on the ith toss} and {get 6 on the ith toss} respectively. Then

$$Cov(X,Y) = \sum_{i,j=1}^{n} Cov(X_i, Y_j) = nCov(X_1, Y_1) = -\frac{n}{18},$$

$$Var(X) = nVar(X_1) = \frac{2n}{9}, \text{ and } Var(Y) = nVar(Y_1) = \frac{5n}{36}.$$

Therefore, the correlation coefficient of X and Y is $-\frac{1}{\sqrt{10}}$.

(15) 4. Suppose the random variables X_n satisfy $EX_n = 0$, $EX_n^2 \leq 1$ and $Cov(X_n, X_m) \leq 0$ for $|m-n| > 2$. Let

$$Y_n = \frac{X_1 + \cdots + X_n}{n}$$

(a) Find a good upper bound for var(Y_n).

Solution: By the Schwarz inequality, $Cov(X_n, X_m) \leq 1$ for all n, m. Therefore,

$$Var(Y_n) = \frac{1}{n^2} \sum_{i,j=1}^{n} Cov(X_i, X_j) \leq \frac{5n-6}{n^2}.$$

(b) Show that Y_n converges to 0 in probability.

Solution: By Chebyshev's inequality,

$$P(|Y_n| > \epsilon) \leq \frac{5n-6}{n^2\epsilon^2} \to 0$$

as $n \to \infty$ for $\epsilon > 0$.

(15) 5. (a) Prove the following: If $\sum_n E|X_n| < \infty$, then $X_n \to 0$ a.s.

Solution: Since $|X_n| \geq 0$, the following interchange is OK:

$$E \sum_n |X_n| = \sum_n E|X_n| < \infty.$$

Therefore $\sum_n |X_n| < \infty$ a.s. If a series converges, then the summands tend to 0. Therefore, $X_n \to 0$ a.s.

(b) Suppose X_n is normally distributed with mean 0 and variance $1/\sqrt{n}$. Then $X_n \to 0$ a.s.

Solution: Since X_n has the same distribution as $X_1/n^{1/4}$, $EX_n^6 = EX_1^6/n^{3/2}$. Therefore, $\sum_n EX_n^6 < \infty$. By part (a), $X_n^6 \to 0$ a.s., so that $X_n \to 0$ a.s.

(15) 6. Recall that the Gamma density with parameters $\alpha > 0$ and $\lambda > 0$ is given by

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0.$$

(a) Find its moment generating function $M(s)$.

Solution:

$$M(s) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-x(\lambda-s)} dx.$$

The integral converges iff $\lambda - s > 0$, and then

$$M(s) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(\frac{y}{\lambda - s}\right)^{\alpha-1} e^{-y} \frac{dy}{\lambda - s} = \left(\frac{\lambda}{\lambda - s}\right)^\alpha.$$

(b) For what values of s is $M(s) < \infty$?

Solution: $s < \lambda$.

(24) 7. Suppose that $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables with distribution function

$$P(X_i \le t) = \begin{cases} \frac{t}{t+1} & \text{for } t \ge 0; \\ 0 & \text{for } t < 0. \end{cases}$$

Let $M_n = \max\{X_1, X_2, \ldots, X_n\}$.

(a) Find the distribution function of M_n.

Solution: For $t \ge 0$,

$$P(M_n \le t) = \left(\frac{t}{t+1}\right)^n = \left(1 - \frac{1}{t+1}\right)^n.$$

(b) Show that $\frac{M_n}{n} \to W$ in distribution for some random variable W.

Solution: For $t > 0$,

$$P\left(\frac{M_n}{n} \le t\right) = P(M_n \le nt) = \left(1 - \frac{1}{nt+1}\right)^n \to e^{-1/t} = P(W \le t)$$

as $n \to \infty$.

(c) What is the distribution of $1/W$?

Solution: For $t > 0$, $P(1/W \ge t) = P(W \le \frac{1}{t}) = e^{-t}$, so $1/W$ is exponential with parameter 1.

(21) 8. Men and women arrive at a store according to independent Poisson processes with parameters 3 and 4 respectively. Men shop for a time that is uniformly distributed on $[0, 1]$, and women shop for a time that is uniformly distributed on $[0, 2]$.

(a) What is the expected number of people who arrive at the store during the interval $[0, 5]$?

Solution: (3+4)5=35.

(b) Given that 6 people arrive during the interval $[2, 3]$, what is the probability that exactly 2 of them are men?

Solution: The conditional distribution of the number of men is $B(6, \frac{3}{7})$. So, the conditional probability is $\binom{6}{2}(\frac{3}{7})^2(\frac{4}{7})^4$.

(c) Given that Sam Smith arrived during the interval $[4, 5]$, what is the probability that he is still in the store at time 5?

Solution: His arrival time is uniform on $[4, 5]$, and the time he stays is uniform on $[0, 1]$. Therefore, the probability that he is still in the store at time 5 is $P(U_1 > U_2) = \frac{1}{2}$, where U_1, U_2 are independent uniforms on $[0, 1]$.

(d) What is the expected number of men in the store at time 5?

Solution: The expected number of men that arrive in $[0, 1]$ is 3, so the answer is $\frac{3}{2}$ by Wald's identity.

(e) What is the expected number of people in the store at time 5?

Solution: By the same argument, the expected number of women in the store at time 5 is $8 \times \frac{1}{2} = 4$. So the expected number of people in the store at time 5 is $\frac{11}{2}$.

(f) What is the probability that there are exactly 3 women in the store at time 5?

Solution: The number of women that arrive during the period $[3,5]$ is Poisson (8). Each will still be in the store at time 5 with probability $\frac{1}{2}$. Therefore, the number that will be in the store at time 5 is Poisson (4), and the probability that there are exactly 3 is $e^{-4}4^3/6$.

(g) What is the probability that the first arrival after time 5 is a woman?

Solution: $\frac{4}{7}$.

(15) 9. Suppose X has the Gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$. (See problem 6 for the density.) Find the density of $Y = \sqrt{X}$.

Solution: For $y > 0$,

$$P(Y \le y) = P(\sqrt{X} \le y) = P(X \le y^2) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{y^2} x^{\alpha-1} e^{-\lambda x} dx.$$

Therefore (don't forget the chain rule!)

$$f_Y(y) = \frac{2\lambda^\alpha}{\Gamma(\alpha)} y^{2\alpha-1} e^{-\lambda y^2}.$$

(15) 10. Suppose $\{X_1, X_2, \dots\}$ is a Bernoulli process with parameter p, and let $N =$ the time at which the second success occurs.

(a) Show that N is a stopping time.

Solution:

$$\{N = n\} = \cup_{k=1}^{n-1} \{X_1 = 0, \dots, X_{k-1} = 0, X_k = 1, X_{k+1} = 0, \dots, X_{n-1} = 0, X_n = 1\},$$

which depends only on $X_1, \dots, X_n$.

(b) Use Wald's equation to compute EN.

Solution: Letting $S_n = X_1 + \cdots + X_n$ as usual, and noting that $S_N = 2$,

$$2 = ES_N = pEN,$$

so $EN = 2/p$.

(13) 11. Let $N(t)$ be a Poisson process with parameter λ, and T be a random variable that is independent of the Poisson process and is exponentially distributed with parameter μ. Find the PMF of $N(T)$, the number of Poisson points in the interval $[0, T]$.

Solution: Let $N_1(t) = N(t)$ and $N_2(t)$ be an independent Poisson process with parameter μ. In the merged process, points belong to N_1 with probability $\frac{\lambda}{\lambda+\mu}$ and to N_2 with probability $\frac{\mu}{\lambda+\mu}$. Therefore,

$$\begin{aligned} P(N(T) = k) &= P(\text{the first } k \text{ points belong to } N_1 \text{ and the next point belongs to } N_2) \\ &= \frac{\lambda^k \mu}{(\lambda+\mu)^{k+1}}. \end{aligned}$$

(15) 12. Use convolutions to compute the density of $Z = X + Y$, where X and Y are independent, with X uniform on $[0, 1]$ and Y exponential with parameter 1.

Solution: For $z > 0$,

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx = \int_0^{z\wedge 1} e^{-(z-x)}dx \\ &= \begin{cases} 1 - e^{-z} & \text{if } 0 < z < 1; \\ e^{1-z} - e^{-z} & \text{if } z > 1. \end{cases} \end{aligned}$$