(25) 1. (a) For a bounded function f on $[a, b]$, define: "f is Riemann integrable on $[a, b]$". (I.e., take Rudin's function to be $\alpha(x) = x$.)

For a partition $P = \{a = x_0 \leq x_1 \leq \cdots \leq x_n = b\}$ of $[a, b]$, let $\Delta x_i = x_i - x_{i+1}$, $m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x)$, $M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x)$, $L(P) = \sum_i m_i \Delta x_i$, and $U(P) = \sum_i M_i \Delta x_i$. Then define

$$\underline{\int} f dx = \sup_P L(P) \quad \text{and} \quad \overline{\int} f dx = \inf_P U(P).$$

By considering common refinements, and since $L(P) \leq U(P)$ for each P,

$$\underline{\int} f dx \leq \overline{\int} f dx. \tag{1}$$

The function f is integrable if these two quantities agree.

(b) Prove directly from the definition that if f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.

Since f is continuous on $[a, b]$, it is uniformly continuous there. Given $\epsilon > 0$, choose $\delta > 0$ so that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. If P is any partition with $\Delta x_i < \delta$ for each i, then $M_i - m_i \leq \epsilon$ for each i. It follows that $U(P) - L(P) \leq \epsilon(b - a)$, so that

$$0 \leq \overline{\int} f dx - \underline{\int} f dx \leq \epsilon(b - a).$$

Since ϵ is arbitrary, equality holds in (1).

(21) 2. Compute $\int_{-1}^{1} (x^2 + 3) d\alpha$, where

$$\alpha(x) = \begin{cases} x & \text{if } x < 0; \\ x^2 + 1 & \text{if } x \geq 0. \end{cases}$$

$$\int_{-1}^{1} (x^2 + 3) d\alpha = \int_{-1}^{0} (x^2 + 3) dx + 3 + \int_{0}^{1} (x^2 + 3)(2x) dx = \frac{10}{3} + 3 + \frac{7}{2} = \frac{59}{6}.$$

(15) 3. Does the series

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

converge uniformly on $[0, 1)$? Explain.

It converges pointwise, but not uniformly. To see this, write for $n \geq 1$

$$\sup_{0 \leq x < 1} \left| \sum_{k=n}^{\infty} (-1)^k x^{2k} \right| = \sup_{0 \leq x < 1} \frac{x^{2n}}{1+x^2} = \frac{1}{2},$$

which does not tend to 0 as $n \to \infty$.

(21) 4. Decide whether each statement below is true or false. If true give a proof; if false give a counterexample, or otherwise disprove it.

(a) If $f_n \to f$ uniformly on $(0,1)$, then $f_n^2 \to f^2$ uniformly on $(0,1)$.

False. Counterexample:

$$f(x) = \frac{1}{x}, \quad f_n(x) = \frac{1}{x} + \frac{1}{n}.$$

Then

$$\sup_{0<x<1} |f_n(x) - f(x)| = \frac{1}{n} \to 0; \sup_{0<x<1} |f_n^2(x) - f^2(x)| = \sup_{0<x<1} \left(\frac{2}{xn} + \frac{1}{n^2} \right) = \infty.$$

(b) If f_n and f are Riemann integrable on $[0,1]$ and $f_n \to f$ pointwise on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 f_n dx = \int_0^1 f dx.$$

False. Counterexample: $f = 0$ and

$$f_n(x) = \begin{cases} n & \text{if } 0 < x < 1/n; \\ 0 & \text{otherwise.} \end{cases}$$

(c) The function

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{5/2}}$$

has a continuous derivative.

True. Let

$$f_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k^{5/2}}$$

be the partial sums of the series. Then f_n is differentiable, and

$$f_n'(x) = \sum_{k=1}^{n} \frac{\cos kx}{k^{3/2}}$$

converges uniformly to

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k^{3/2}}.$$

Since $f_n(0) = 0$ for each n, the result follows from Theorem 7.17.

(18) 5. In each case, state whether the assertion is true or false. No explanation is needed.

(a) The family of functions $\{f_n(x) = \sin nx, n = 1, 2, \dots\}$ is equicontinuous on $[0, 1]$.

False, since

$$\sup_{|x-y|\le \pi/n} |f_n(x) - f_n(y)| = 2.$$

(b) If $A \subset [0, 1]$ is countable, the indicator function 1_A is Riemann integrable on $[0, 1]$.

False; take $A = Q \cap [0, 1]$.

(c) If $f_n(0) = 0$ and f_n' exists, is continuous, and converges to 0 uniformly on $[-1, 1]$, then f_n converges to 0 uniformly on $[-1, 1]$.

True, by Theorem 7.17.

(d) If $|f_n(x)| \le 1$ for all n and x, and $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[0, 1]$, then $\sum_{n=1}^{\infty} \sup_{0\le x\le 1} |f_n(x)| < \infty$.

False. Counterexample:

$$f_n(x) = \begin{cases} 1/n & \text{if } 1/(n+1) < x < 1/n; \\ 0 & \text{otherwise.} \end{cases}$$

(e) If $f \in \mathcal{R}(\alpha)$ and g is continuous, then $h \in \mathcal{R}(\alpha)$, where $h(x) = g(f(x))$.

True, by Theorem 6.11.

(f) If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and $\epsilon > 0$, then there is a function $g \in C[a, b]$ so that $||f - g|| < \epsilon$, where $||h|| = \sup_{a\le x\le b} |h(x)|$.

False; take $a = 0, b = 1$ and

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

If $g \in C[0, 1]$, the $||f - g|| \ge \frac{1}{2}$.