(15) 1. For which real values of $x$ does the series

$$\sum_{n=1}^{\infty} \frac{(nx)^n}{n!}$$

converge? Explain.

By Stirling’s formula, $n! \sim n^n e^{-n} \sqrt{2\pi n}$, so

$$\frac{(nx)^n}{n!} \sim \frac{(xe)^n}{\sqrt{2\pi n}}.$$

It follows that the series converges for $|x| < e^{-1}$, and diverges for $|x| > e^{-1}$ and for $x = e^{-1}$. The series converges for $x = -e^{-1}$ by the alternating series test. To check this, one needs to know that

$$a_n = \frac{(ne^{-1})^n}{n!}$$

is decreasing to zero. The inequality $a_n \geq a_{n+1}$ is equivalent to $n \log(1 + \frac{1}{n}) \leq 1$. This follows from $\log(1 + x) \leq x$ for $x \geq 0$. (Note: $a_n \sim 1/\sqrt{n}$ does not imply that $\sum (-1)^n a_n$ converges. For an example in which it diverges, take $a_n = 1/\sqrt{n}$ for $n$ odd and $a_n = (1/\sqrt{n}) + (1/n)$ for $n$ even.)

(15) 2. Use power series to compute

$$\lim_{x \to 0} \frac{a^x - 1}{x}$$

for $a > 0$.

For $x \neq 0$, write

$$\frac{a^x - 1}{x} = \frac{e^{x \log a} - 1}{x} = \log a \sum_{n=1}^{\infty} \frac{(x \log a)^{n-1}}{n!}.$$

Therefore, the limit is $\log a$.

(20) 3. Suppose $f$ is a real-valued function on $[-1, 1]$ with a continuous derivative. Prove that there exist polynomials $p_n(x)$ so that $p_n \to f$ and $p'_n \to f'$ uniformly on $[-1, 1]$. (You may use the Weierstrass Theorem.)

Choose polynomials $q_n$ so that $q_n \to f'$ uniformly on $[-1, 1]$, and let

$$p_n(x) = f(-1) + \int_{-1}^{x} f'(t) dt.$$

(20) 4. Suppose $f$ is Riemann integrable on compact subsets of $[0, \infty)$.
(a) Show that \( \lim_{x \to \infty} f(x) = 0 \) implies
\[
\lim_{t \downarrow 0} t \int_0^\infty e^{-tx} f(x) \, dx = 0.
\]

Given \( \epsilon > 0 \), choose \( M \) so that \( |f(x)| < \epsilon \) for \( x \geq M \). Then
\[
\left| t \int_0^\infty e^{-tx} f(x) \, dx \right| \leq \epsilon \int_M^\infty t e^{-tx} \, dx + \|f\|_\infty \int_0^M t e^{-tx} \, dx = \epsilon e^{-Mt} + \|f\|_\infty [1 - e^{-Mt}].
\]
Therefore,
\[
\lim_{t \downarrow 0} \left| t \int_0^\infty e^{-tx} f(x) \, dx \right| \leq \epsilon.
\]
Since \( \epsilon \) is arbitrary, the limit is 0.

(b) Is the converse to the statement in part (a) true? If so, prove it; if not give a counterexample.

The converse is not true. For a counterexample, take \( f(x) = (-1)^n \) on \([n, n+1)\). Then
\[
t \int_0^\infty e^{-tx} f(x) \, dx = \sum_{n=0}^{\infty} (-1)^n \int_n^{n+1} t e^{-tx} \, dx = (1 - e^{-t}) \sum_{n=0}^{\infty} (-e^{-t})^n = \frac{1 - e^{-t}}{1 + e^{-t}}.
\]

(15) 5. Change the order of summation, with appropriate justification, to compute
\[
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{k+l}{k} x^k y^l \quad \text{for } |x| + |y| < 1.
\]

If \( x, y \geq 0 \), the following interchange is justified, whether or not the series converges:
\[
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{k+l}{k} x^k y^l = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{n=0}^{\infty} (x + y)^n.
\]
Therefore,
\[
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{k+l}{k} |x|^k |y|^l = \frac{1}{1 - |x| - |y|} < \infty \quad \text{for } |x| + |y| < 1.
\]
It follows that the interchange in justified whenever $|x| + |y| < 1$, and this gives

$$
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{k+l}{k} x^k y^l = \frac{1}{1 - x - y}
$$

in this case.

(15) 6. Suppose that $f_n$ is continuous on $E$ for each $n$ and $f_n \to f$ uniformly on $E$. Prove that $f$ is continuous on $E$.

Begin by writing

$$
|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|.
$$

To prove continuity at $x \in E$, given $\varepsilon > 0$, choose $n$ so that $|f_n(z) - f(z)| < \varepsilon$ for all $z \in E$, and then $\delta > 0$ so that $d(y, x) < \delta$ implies $|f_n(y) - f_n(x)| < \varepsilon$ for that $n$. Then $d(y, x) < \delta$ implies $|f(y) - f(x)| < 3\varepsilon$.

(20) 7. Suppose that for each $n$, $f_n$ is continuous on $[0, 1]$ and satisfies $|f_n(x)| \leq 1$ for $0 \leq x \leq 1$. Define

$$
g_n(x) = \int_0^x f_n(t) dt.
$$

(a) Show that there is a sequence $n_k$ and a continuous function $g$ on $[0, 1]$ so that $g_{n_k} \to g$ uniformly on $[0, 1]$.

Since $|g_n(y) - g_n(x)| \leq |y - x|$, and $|g_n(x)| \leq 1$ for all $n, x, y$, the family $\{g_n\}$ is uniformly bounded and equicontinuous. The result follows from Theorem 7.25.

(b) Is the function $g$ in part (a) necessarily differentiable on $[0, 1]$? Explain.

No. For a counterexample, let

$$
g_n(x) = \begin{cases} 
|x - \frac{1}{2}| & \text{if } |x - \frac{1}{2}| \geq \frac{1}{n}; \\
\frac{n}{2}(x - \frac{1}{2})^2 + \frac{1}{2n} & \text{if } |x - \frac{1}{2}| \leq \frac{1}{n};
\end{cases}
$$

and $f_n = g'_n$.

(20) 8. Suppose $f$ is continuous on $[0, 1]$, and let $||f||_p = \left[ \int_0^1 |f(x)|^p dx \right]^{1/p}$ for $p \geq 1$ and $||f||_\infty = \max_{0 \leq x \leq 1} |f(x)|$. 

(a) Show that $||f||_p \leq ||f||_\infty$ for each $p$.

$$||f||_p \leq \left[ \int_0^1 ||f||_\infty^p dx \right]^{1/p} = ||f||_\infty.$$  

(b) Show that $\lim_{p \to \infty} ||f||_p = ||f||_\infty$.

Without loss of generality, we may assume that $||f||_\infty > 0$. Take $M \in (0, ||f||_\infty)$, and then an interval $(a, b) \subset [0, 1]$ (with $b > a$) so that $|f(x)| \geq M$ on $(a, b)$. Then

$$||f||_p \geq \left[ \int_a^b |f(x)|^p dx \right]^{1/p} \geq M(b - a)^{1/p}.$$  

It follows that

$$\liminf_{p \to \infty} ||f||_p \geq M.$$  

Now take $M$ close to $||f||_\infty$, and use part (a).

(30) 9. In each case, decide whether the statement is true or false. If true, prove it; if false, give a counterexample, or otherwise show it is false.

(a) If $f_n$ and $f$ are Riemann integrable on $[0, 1]$ and $f_n \to f$ uniformly on $[0, 1]$, then $\int_0^1 f_n dx \to \int_0^1 f dx$ as $n \to \infty$.

True:

$$\left| \int_0^1 f_n dx - \int_0^1 f dx \right| \leq ||f_n - f||_\infty \to 0.$$  

(b) If $\sum_n a_n e^{inx}$ is the Fourier series for the function

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq \pi; \\ x - 2\pi & \text{for } \pi < x \leq 2\pi, \end{cases}$$  

then $\sum_n |a_n| < \infty$.

False. If it were true, then $\sum_n a_n e^{inx}$ would converge uniformly to a continuous function $g$. But $g = f$ for $x \neq \pi$, since $f$ is differentiable there. This is a contradiction, since $f$ has a jump discontinuity at $\pi$.

(c) Every bounded function on $[0, 1]$ is Riemann integrable.

False. Counterexample: the indicator of the rationals.
(d) If $f$ in a continuous complex-valued function on $[0, 1]$, then there exists a $t \in [0, 1]$ so that
\[ \int_0^1 f(x)dx = f(t). \]

False. Counterexample: $f(x) = e^{2\pi ix}$.

(30) 10. In each case, decide whether the statement is true or false. If true, prove it; if false, give a counterexample, or otherwise show it is false.

(a) If the power series $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence 1, then $\sum_{n=0}^{\infty} c_n$ converges.

False. Counterexample: $c_n = 1/n$.

(b) The space $C[0, 1]$ of continuous functions on $[0, 1]$ with the norm $\| \cdot \|_2$ is complete.

False. Take $f_n(x) = \min\{(2x)^n, 1\}$ and $f = 1_{[1/2,1]}$. Then $f_n \rightarrow f$, so $\{f_n\}$ is Cauchy. If $f_n \rightarrow g$ for some continuous $g$, then $\|f - g\|_2 = 0$. Since $f - g$ is continuous except at $1/2$, $f = g$ except at $1/2$. This is a contradiction.

(c) \[ \frac{d}{dx} \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2} = -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1 + nx^2)^2}. \]

True. The series on the right converges uniformly on compact sets, so the statement follows from Theorem 7.17, applied to the partial sums.

(d) If $f$ is nonnegative and continuous on $[0, \infty)$, and satisfies $\lim_{x \rightarrow \infty} f(x) = 0$, then
\[ \int_0^{\infty} f(x)dx < \infty \text{ if and only if } \sum_{n=1}^{\infty} f(n) < \infty. \]

False. Take $f(x) = (x - n)(1 - x + n)/(n + 1)$ for $n \leq x \leq n + 1$. 

\[ f(x) = (x - n)(1 - x + n)/(n + 1) \text{ for } n \leq x \leq n + 1. \]