(15) 1. (a) Define: "x is a limit point of E".

For every $\epsilon > 0$ there is a $y \neq x, y \in E$ so that $d(x, y) < \epsilon$.

Let E' be the set of limit points of E. Decide whether each of the following statements is true or false. If true, prove it; if false, give a counterexample.

(b) $(E \cap F)' \subset E' \cap F'$

True. Suppose $x \in (E \cap F)'$. To show $x \in E'$, take $\epsilon > 0$. Since $x \in (E \cap F)'$, there is a $y \neq x, y \in E \cap F$ so that $d(x, y) < \epsilon$. This y is in E. So, $x \in E'$. Similarly, $x \in F'$.

(c) $(E \cap F)' \supset E' \cap F'$

False. If $X = R^1$, take $E = (0, 1)$ and $F = (1, 2)$. Then $(E \cap F)' = \emptyset$ and $E' \cap F' = \{1\}$.

(15) 2. Let $\mathcal{N} = \{1, 2, ...\}$ be the set of natural numbers, $\mathcal{F}$ be the collection of finite subsets of $\mathcal{N}$, and $\mathcal{S}$ be the collection of all subsets of $\mathcal{N}$.

(a) Prove that $\mathcal{F}$ is countable.

$\mathcal{F} = \cup_{n=0}^{\infty} \mathcal{F}_n$, where $\mathcal{F}_n = \{A \subset \mathcal{N} : |A| = n\}$. So, it suffices to show that $\mathcal{F}_n$ is countable for $n \geq 1$. $\mathcal{N}^n$ is countable, so

$$\{(a_1, ..., a_n)\} \in \mathcal{N}^n : a_1 < a_2 < \cdots < a_n\}$$

is countable. But this is one-to-one correspondence with $\mathcal{F}_n$.

(b) Find a one-to-one correspondence between $\{0, 1\}^{\mathcal{N}}$ and $\mathcal{S}$.

Define $f : \{0, 1\}^{\mathcal{N}} \to \mathcal{S}$ by $f(x_1, x_2, ...) = \{k \in \mathcal{N} : x_k = 1\}$.

(c) Is $\mathcal{S}$ countable or uncountable? Why?

It is uncountable, since $\{0, 1\}^{\mathcal{N}}$ is uncountable.

(15) 3. Decide whether each of the following statements is true or false. No explanation is needed.

(a) Q, the set of rationals, has the least upper bound property. False

(b) The Cantor set contains both rational and irrational points. True

(c) Every metric space has at most finitely many subsets that are both open and closed. False

(d) Every subset of a general metric space that is closed and bounded is compact. False

(e) The union of two compact sets is compact. True

(15) 4. (a) Define "K is compact".

Every open cover has a finite subcover.

(b) Prove that every compact set is bounded.

Fix and $x \in X$ and let $\mathcal{O}_n = \{y \in X : d(x, y) < n\}$. This is an open cover of the compact set K, and the $\mathcal{O}_n$'s are increasing, so there is an n so that $K \subset \mathcal{O}_n$. Therefore K is bounded.

(c) Prove directly from the definition that $[0, 1)$ is not compact in R^1.

$\{(-1, 1 - \frac{1}{n}), n \geq 1\}$ is an open cover of $[0, 1)$ that has no finite subcover.

(20) 5. For each part, give an example of a set A in R^1 with the usual metric that has the required properties. There is no need to prove that it has those properties:

(a) A is countable and compact.

$A = \{0\} \cup \{\frac{1}{n} : n \geq 1\}$.

(b) A and A^c are both dense.

$A = Q$.

(c) A is not connected, but $\overline{A}$ is.

$(0, 1) \cup (1, 2)$.

(d) A is countable and has no limit points.

$A = \mathcal{N}$

(10) 6. Suppose $A \subset R^1$ is uncountable. Prove that A has a limit point.

$A = \cup_{n=-\infty}^{\infty} A \cap [n, n+1]$, so there is an n for which $A \cap [n, n+1]$ is infinite. This is an infinite subset of the compact set $[n, n+1]$, so has a limit point.

(10) 7. Show that for each x, $\mathcal{O} = \{y \in X : d(x, y) > 1\}$ is open.

Suppose $y \in \mathcal{O}$. Let $\epsilon = d(x, y) - 1 > 0$. By the triangle inequality, if $d(z, y) < \epsilon$, then

$$d(x, y) \leq d(x, z) + d(z, y) < d(x, z) + \epsilon,$$

so $d(x, z) > 1$. Therefore, the ϵ neighborhood of y is contained in $\mathcal{O}$.