T. Liggett Mathematics 131AH – Final Exam December 9, 2009

1	2	3	4	5	6	7	Total

8	9	10	11	12	13	14	Total

Last name: First name:

(15) 1. (a) Prove the following part of the ratio test: If $a_n \neq 0$ for each n and

$$\limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1,$$

then $\sum_n a_n$ converges absolutely.

(b) Show by example that the statement in (a) is false if lim sup is replaced by lim inf.

(20) 2. Suppose $f : X \to Y$ is continuous. In each case, decide whether the statement is true or false. If true, prove it; if false give a counterexample with $X = Y = R^1$.

(a) If $K \subset X$ is compact, then $f(K)$ is compact.

(b) If $K \subset Y$ is compact, then $f^{-1}(K)$ is compact.

(c) If $E \subset X$ is connected, then $f(E)$ is connected.

(d) If $E \subset Y$ is connected, then $f^{-1}(E)$ is connected.

(28) 3. In each case, say whether the statement is true or false. Briefly explain your answer.

(a) If $\{a_n, n \geq 1\}$ is decreasing and $\sum a_n$ converges, then there exists a constant C so that $a_n \leq C/n$.

(b) Let X be $C[0,1]$, the metric space of all continuous functions on $[0,1]$, with $d(f,g) = \max_{0\leq t\leq 1} |f(t) - g(t)|$. Then $\{f \in X : d(f,0) \leq 1\}$ is compact.

(c) If f is continuous on $(0,1)$, it is uniformly continuous on $(0,1)$.

(d) $Q \cap [0,1]$ is compact.

(e) If $\sum_n a_n$ converges and $\{b_n\}$ is bounded, then $\sum_n a_n b_n$ converges.

(f) If a_n and b_n are real and $\sum_n (a_n^2 + b_n^2) < \infty$, then $\sum_n a_n b_n$ converges.

(g) If $\sum_n |a_{n+1} - a_n| < \infty$, then $\lim_{n\to\infty} a_n$ exists.

(15) 4. A family $\mathcal{F}$ of functions is said to be uniformly equicontinuous if

$$\forall\, \epsilon > 0\ \exists\, \delta > 0 \ni d(x,y) < \delta, f \in \mathcal{F} \Rightarrow d(f(x), f(y)) < \epsilon.$$

(a) Suppose $g : R^2 \to R^1$, and define $f_\theta : R^1 \to R^1$ by $f_\theta(x) = g(x\cos\theta, x\sin\theta)$ for $0 \le \theta \le 2\pi$. Prove that if $\{f_\theta, 0 \le \theta \le 2\pi\}$ is uniformly equicontinuous, then g is continuous at the origin.

(b) Show by example that the statement in (a) is false if it is only assumed that each f_θ is continuous.

(10) 5. Prove that if $\lim_{n\to\infty} a_n = a$, then

$$\lim_{x\uparrow 1}(1-x)\sum_{n=0}^{\infty} x^n a_n = a.$$

(12) 6. (a) Suppose that $F, K \subset X$, $F \cap K = \emptyset$, F is closed and K is compact. Show that $\inf\{d(x,y) : x \in F, y \in K\} > 0$.

(b) Show by example that the statement in (a) is not correct if K is only assumed to be closed, rather than compact.

(15) 7. Suppose $a < c < b$, f in continuous on (a,b), and f is differentiable on $(a,b)\backslash\{c\}$. Show that if $\lim_{x \to c} f'(x)$ exists, then f is differentiable at c also.

(10) 8. Suppose f is a nonnegative function on R^1 such that for some M,

$$\sum_{x \in F} f(x) \leq M$$

for all finite $F \subset R^1$. Show that $\{x : f(x) > 0\}$ is at most countable.

(10) 9. Is Q, the set of rational numbers, connected? Prove your answer.

(15) 10. (a) Define $f : R^1 \to R^1$ is differentiable at x.

(b) Prove that if f is differentiable at x, then it is continuous at x.

(c) Prove that if f and g are differentiable at x, then so is their product fg.

(10) 11. Suppose that f is strictly positive and continuous on $[0, \infty)$, and that $\lim_{x \to \infty} f(x) = 1$. Show that there is an $\epsilon > 0$ so that $f(x) \geq \epsilon$ for all $x \geq 0$.

(10) 12. Suppose that $f : [0,1] \to R^1$ is continuous and satisfies $f(0) = f(1) = 0$ and $f'(0) = f'(1) = 1$. (f may not be differentiable on $(0,1)$.) Show that there is an $x \in (0,1)$ so that $f(x) = 0$.

(10) 13. Show that the sequence x_n defined by $x_1 = 1$ and

$$x_{n+1} = x_n + \frac{1}{x_n^2}, \quad n \geq 1$$

is unbounded.

(20) 14. Suppose $a_n \downarrow 0$ and $\sum_n a_n = \infty$.

(a) Determine exactly for which complex z's the series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges absolutely.

(b) Determine exactly for which complex z's the series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges.