Review, VaR/TVaR, and Probability Theory

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Math 178B, Week 9

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Overview

1. Content Review

2. Problem Solving
Announcements

- See Gradescope for your Quiz 4 grades.
- Today will be our final discussion section.
- As always, office hours are on an appointment basis. Please send me an email if you’d like to meet.
Value at Risk (VaR) is generally the amount of capital required to ensure that with a high degree of certainty, the policy provider will not become insolvent. We have already seen this quantity called quantile risk. See KPW page 43 for a full description of this.

- The VaR of a random variable (usually a loss) at the 100p% level is the 100p-th percentile of X’s distribution, which mathematically is

\[
VaR_p(X) = \inf_{x \geq 0} \{ x \mid F_X(x) \geq p \}, \quad 0 < p < 1.
\]

- If X is continuous at \( VaR_p(X) \), then \( F_X[VaR_p(X)] = p \).
Tail Value at Risk

We have also already seen this value as the $CTE_p(X)$, but it is also called the $TVaR_p(X)$. This quantity is the average of all VaR values above the level $p$.

- This can be written as a conditional expectation,

$$TVaR_p(X) = \mathbb{E}[X \mid X > \text{VaR}_p(X)].$$

- Alternatively, this may be written as

$$TVaR_p(X) = \frac{\int_p^1 \text{VaR}_u(X) \, du}{1 - p}.$$

- Sometimes, it is said that this risk measure provides a definition of “bad times”, i.e. when a rare downturn or insurance event occurs.
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Questions?
Content Review

Problem Solving
Problem 1

Consider an exponential random variable $X$ with mean $\theta$ and pdf

$$f(x) = \frac{1}{\theta} \exp \left( -\frac{x}{\theta} \right), \quad x > 0.$$  \hspace{1cm} (1)

Calculate the $\text{VaR}_p(X)$ and $\text{TVaR}_p(X)$. 
First we need to find the cumulative probability distribution, denoted by $F_X(x)$. This can be done by integrating,

$$F_X(x) = \int_{-\infty}^{x} f_X(s) \, ds$$

$$= \frac{1}{\theta} \int_{0}^{x} \exp \left( \frac{-s}{\theta} \right) \, ds$$

$$= -\frac{1}{\theta} \theta \exp \left( \frac{-s}{\theta} \right) \bigg|_{0}^{x}$$

$$= 1 - \exp \left( \frac{-x}{\theta} \right).$$
Then, denoting the Value at Risk of $X$ to the 100$p\%$ level as $\pi_p$, by definition this is

$$F(\pi_p) = p.$$  

This can then be found by simple algebra,

$$1 - e^{-\pi_p/\theta} = p$$

$$\frac{-\pi_p}{\theta} = \ln(1 - p)$$

$$\pi_p = -\theta \ln(1 - p).$$
To find the Tail Value at Risk, we may use the equation given on page 45 since $X$ is a continuous random variable,

$$TVaR_p(X) = \mathbb{E}[X \mid X > \text{VaR}_p(X)].$$

Recalling the equation for conditional expectation, see Computation at https://en.wikipedia.org/wiki/Conditional\_expectation, this can be written as

$$TVaR_p(X) = \frac{\int_{\pi_p}^{\infty} x f_X(x) \, dx}{1 - p}$$
Problem 1

Continuing this computation,

\[ TVaR_p(X) = \frac{\int_{\pi_p}^{\infty} x f_X(x) \, dx}{1 - p} \]

\[ = \frac{\int_{\pi_p}^{\infty} \frac{x}{\theta} e^{-x/\theta} \, dx}{1 - p} \]

\[ = \left[ -\frac{x e^{-x/\theta}}{\theta} \right]_{\pi_p}^{\infty} + \frac{1}{1 - p} \int_{\pi_p}^{\infty} e^{-x/\theta} \, dx \]

\[ = \frac{\pi_p e^{-\pi_p/\theta} + \left[ -\theta e^{-x/\theta} \right]_{\pi_p}^{\infty}}{1 - p} \]

\[ = \frac{\pi_p (1 - p) + \theta e^{-\pi_p/\theta}}{1 - p} \]

\[ = \pi_p + \theta = -\theta \ln(1 - p) + \theta. \]
Problem 2

This problem gives practice on conditional probability and transformations of random variables.
Let \( X \) be uniformly distributed between \([0, 1]\), and define

\[
Y = X^2, \quad Z = \begin{cases} 
0, & 0 \leq X \leq \frac{1}{2} \\
1, & \frac{1}{2} < X \leq 1 
\end{cases}
\]  \hspace{1cm} (2)

Find \( E[Y|Z] \).
Begin by noticing that $Z$ only takes two values, and is therefore discrete. We only need to compute $\mathbb{E}[Y|Z = 0]$ and $\mathbb{E}[Y|Z = 1]$. To do so, translate the conditional probability to using the variable $X$ only, as follows.

$$
\mathbb{E}[Y|Z = 0] = \mathbb{E}[X^2|0 \leq X \leq \frac{1}{2}]
$$

$$
= \int_{0}^{1/2} x^2 \, dx
$$

$$
= \frac{x^3}{3} \bigg|_{0}^{1/2}
$$

$$
= \frac{1}{12}.
$$
To compute $\mathbb{E}[Y|Z = 1]$, do more or less the same thing,

$$\mathbb{E}[Y|Z = 1] = \mathbb{E}[X^2|\frac{1}{2} < X \leq 1]$$

$$= \int_{1/2}^{1} x^2 \, dx$$

$$= \frac{1}{3} \bigg|_{1/2}^{1} \frac{1}{2} = \frac{7}{12}.$$

Therefore, one has that

$$\mathbb{E}[Y|Z] = \frac{1}{2}Z + \frac{1}{12} = \begin{cases} \frac{1}{12} & Z = 0 \\ \frac{7}{12} & Z = 1 \end{cases}.$$
Assume you roll a fair die, and win money according to the following rule:

- Roll a 1-3 - receive nothing
- Roll a 4 or 5 - receive $1
- Roll a 6 - receive $3

Suppose you are allowed to roll the die repeatedly until you get a 6. How much should the house charge for someone to play this game in order to be profitable in the long run, assuming no other fees/cheating?

Let $Y$ be the total amount of winnings for one instance of the game.
Solution

Letting $N$ be the number of the roll a 6 appears, $X_i$ be the $i$-th’s game’s roll, and $Y_i$ be the corresponding winnings, calculating $\mathbb{E}[Y|N]$ will be useful for calculating the expected total winnings per game. For example, if $N = n$, we know $X_1, \ldots, X_{n-1}$ were not a 6, and thus uniformly distributed on $\{1, 2, 3, 4, 5\}$. Therefore

$$\mathbb{E}[Y_i|N = n] = 0 \times \Pr(Y_i = 0|N = n) + 1 \times \Pr(Y_i = 0|N = n) = \frac{2}{5}$$
Furthermore, one knows that $E[Y_n|N = n] = 3$, since this ends the game. Therefore, one obtains

$$E[Y|N = n] = E[Y_1 + \cdots + Y_n|N = n]$$

$$= (n - 1) \cdot \frac{2}{5} + 3$$

$$= \frac{2}{5} n + \frac{13}{5}.$$

This means that $E[Y|N] = \frac{2}{5} N + \frac{13}{5}$. Finally, to find $E[Y]$, we must do the expectation again, meaning

$$E[Y] = E[E[Y|N]] = E \left[ \frac{2}{5} N + \frac{13}{5} \right] = \frac{2}{5} E[N] + \frac{13}{5}.$$
Problem 3

Solution

Given that

\[ \mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|N]] = \mathbb{E} \left[ \frac{2}{5}N + \frac{13}{5} \right] = \frac{2}{5} \mathbb{E}[N] + \frac{13}{5}, \]

we only need to find \( \mathbb{E}[N] \). The number of games \( N \) is geometrically distributed, meaning that \( f_N(n) = (1 - p)^{n-1}p \), or \( n-1 \) successes followed by a failure. One can show that a geometric sum has the expectation of \( \frac{1}{p} \), and \( p \) here is \( \frac{1}{6} \). This means that \( \mathbb{E}[Y] = \frac{2}{5} \cdot 6 + \frac{13}{5} = 5 \). The house should charge more than $5 per game to be profitable in the long run.