Overview

1. Content Review

2. Problem Solving
Announcements

- See Gradescope for your Quiz 1 grades
- Regrades must be requested on Gradescope
- In the future, points will be discounted when not assigning pages to problems
- Office Hours this week are 1:30 - 3PM Wednesday
Recall some essentials

- For whole life insurance where a benefit is to be paid out at the time of an unknown time of death, the Expected Present Value (EPV) of a $1 insurance benefit is denoted $A_x$
- The EPV of a whole life (continuous) annuity is related to the former value, as
  \[
  \overline{a}_x = \frac{1 - \overline{A}_x}{\delta} = \int_0^\infty e^{-\delta t} t p_x \, dt
  \]
  where $\delta$ is the continuous compounding interest rate
- A classic way of calculating premiums is derived from the Equivalence Principle, which states that the EPV of premium income = EPV of benefit provided (Section 6.5)
Content Review

Problem Solving
Problem Solving

The following three questions were covered during discussion.
Problem 1

Consider a multi-state model with states Healthy (0), Disabled (1), and Dead (2). For $k = 0$ and 1, we are provided the probabilities

$$p_{x+k}^0 = 0.7, \quad p_{x+k}^1 = 0.2, \quad p_{x+k}^{10} = 0.1, \quad p_{x+k}^{12} = 0.25.$$ 

At $t = 0$, we have 100 healthy policyholders whose health statuses are independent of one another.

**Problem**

*Calculate the expected value and variance of the number of these original 100 who die within the first two years.*
Problem 1

Recall,

\[ p^{00}_{x+k} = 0.7, \ p^{01}_{x+k} = 0.2, \ p^{10}_{x+k} = 0.1, \ p^{12}_{x+k} = 0.25. \]

Displayed graphically, one has

\[ \begin{align*}
0.7 & \quad 0.2 & \quad 0.25 \\
0.1 & \quad 0.65 & \quad 1.0 \\
1 & & 2
\end{align*} \]

**Figure:** The three-state Markov chain describing the transitions between healthy (0), disabled (1), and dead (2) states. Missing from this image is a 0.1 transition probability from (0) to (2).
Problem 1

Problem

Calculate the expected value and variance of the number of these original 100 who die within the first two years.

Solution

Let $X_i$ be the random variable such that $X_i = 1$ or $X_i = 0$ if the policyholder $i$ is or is not in state “2” after 2 years, respectively. We want to know $\mathbb{E}[X_i]$ and $\text{Var}(X_i)$ since

$$\mathbb{E} \left( \sum_{i=1}^{100} X_i \right) = 100 \mathbb{E}[X_1]$$
$$\text{Var} \left( \sum_{i=1}^{100} X_i \right) = 100 \text{Var}[X_1]$$
Problem 1

To calculate $E[X]$ and $\text{Var}(X)$, we can use the formulae

$$E[X] = \sum_{x \in \{0, 1\}} xp(x) = 0(0.85) + 1(0.15) = 0.15.$$ 

Thus, under this model we expect 5 deaths after 2 years. To obtain the variance, compute

$$\text{Var}[X] = \sum_{x \in \{0, 1\}} p(x)(x - 0.05)^2 = (0.85)(0 - 0.15)^2 + (0.15)(1 - 0.15)^2.$$
Details I omitted:

- I determined $p(x = 0) = 0.85$ and $p(x = 1) = 0.15$, how?
- I didn’t state why

$$\text{Var} \left( \sum_{i=1}^{100} X_i \right) = 100 \text{Var}[X_1],$$

Can you answer why?

- I filled in extra values in the Markov chain. Were they necessary? Could you have obtained them on your own?
Exercise 8.20 A husband and wife, aged 65 and 60 respectively, purchase an insurance policy, under which the benefits payable on first death are a lump sum of $10,000, payable immediately on death, plus an annuity of $5,000 per year payable continuously throughout the lifetime of the surviving spouse. A benefit of $1,000 is paid immediately on the second death. Premiums are payable continuously until the first death.

You are given that $\ddot{A}_{60} = 0.353789$, $\ddot{A}_{65} = 0.473229$ and that $\ddot{A}_{60:65} = 0.512589$ at 4% per year effective rate of interest. The lives are assumed to be independent.

(a) Calculate the EPV of the lump sum death benefits, at 4% per year interest.
(b) Calculate the EPV of the reversionary annuity benefit, at 4% per year interest.
(c) Calculate the annual rate of premium, at 4% per year interest.
(d) Ten years after the contract is issued the insurer is calculating the policy value.
   (i) Write down an expression for the policy value at that time assuming that both lives are still surviving.
   (ii) Write down an expression for the policy value assuming that the husband has died but the wife is still alive.
   (iii) Write down Thiele’s differential equation for the policy value assuming (1) both lives are still alive, and (2) only the wife is alive.
Solution (a)

To calculate the EPV of the lump sum death benefits, note that one could view this as $1000 payable at each death plus $9000 payable on the first death. The expected benefit first death payout is described by $A_{60:65}$ and thus,

\[
1000A_{60} + 1000A_{65} + 9000A_{60:65} = 5440.32.
\]  

(1)
Recall that a *reversionary annuity* is an annuity that starts payment on the death of one life in a couple, and continues throughout the lifetime of the remaining spouse.

**Solution (b)**

*There is a reversionary annuity to each of the lives, so the EPV is*

\[
5000(\bar{a}_{65} - \bar{a}_{65:60}) + 5000(\bar{a}_{60} - \bar{a}_{65:60}) \\
= 5000\left(\frac{1 - \bar{A}_{65}}{\delta} + \frac{1 - \bar{A}_{60}}{\delta} - 2 \frac{1 - \bar{A}_{60:65}}{\delta}\right) \\
= \$25262.16.
\]

*Note that* \(\delta = \ln(1 + 0.04) \approx 0.03922\).
Solution (c)

To calculate the annual rate of the premium, use the Equivalence Principle equation for the premium $P$,

\[
\text{EPV of premium income} = \text{EPV of benefit} \quad \text{(2)}
\]

\[
P\bar{a}_{65:60} = (5440.32 + 25262.16) \quad \text{(3)}
\]

\[
P = \frac{(5440.32 + 25262.16)}{1 - \overline{A}_{60:65} \delta} \quad \text{(4)}
\]

\[
= \$2470.55 \quad \text{(5)}
\]
Problem 2

Ten years after the contract is issued the insurer is calculating the policy value.

Solution (d)

(i) To write down an expression for the policy value assuming that both lives are still surviving, use the equation for \( 10 V^{(0)} \)

\[
10 V^{(0)} = 5000(\bar{a}_{70} + \bar{a}_{75} - 2\bar{a}_{75:70}) + 1000(\bar{A}_{70} + \bar{A}_{75}) + 9000\bar{A}_{75:70} - P\bar{a}_{75:70}.
\]

(ii) To write down an expression for the policy value assuming that the husband has died but the wife is still alive, use the equation for \( 10 V^{(2)} \)

\[
10 V^{(2)} = 5000\bar{a}_{70} + 1000\bar{A}_{70}.
\]

See section 8.7 for more on this.
Problem 2

Ten years after the contract is issued the insurer is calculating the policy value.

Solution (d)

(iii) To write down Thiele’s differential equation for the policy value assuming (1) both lives are still alive, and (2) only the wife is alive, use the equation (8.23) on page 255 of the book:

\[ \frac{d}{dt} t V^{(0)} = \delta_t V^{(0)} + P - \mu_{65+t:60+t}^{02} (t V^{(2)} + 10000 - t V^{(0)}) - \mu_{65+t:60+t}^{01} (t V^{(1)} + 10000 - t V^{(0)}). \]

and for (2),

\[ \frac{d}{dt} t V^{(2)} = \delta_t V^{(2)} - 5000 - \mu_{60+t}^{23} (1000 - t V^{(2)}). \]
Problem 3

Part (a) only:

**Exercise 8.15** Ryan is entitled to an annuity of $100,000 per year at retirement, paid monthly in advance, and the normal retirement age is 65. Ryan’s wife, Lindsay, is two years younger than Ryan.

(a) Calculate the EPV of the annuity at Ryan’s retirement date.
(b) Calculate the revised annual amount of the annuity (payable in the first year) if Ryan chooses to take a benefit which provides Lindsay with a monthly annuity following Ryan’s death equal to 60% of the amount payable whilst both Ryan and Lindsay are alive.
(c) Calculate the revised annual amount of the annuity (payable in the first year) if Ryan chooses the benefit in part (b), with a ‘pop-up’ – that is, the annuity reverts to the full $100,000 on the death of Lindsay if Ryan is still alive. (Note that under a ‘pop-up’, the benefit reverts to the amount to which Ryan was originally entitled.)

Basis:

Male mortality before and after widowerhood:

Makeham’s law, \( A = 0.0001, B = 0.0004 \) and \( c = 1.075 \)

Female survival before widowhood:

Makeham’s law, \( A = 0.0001, B = 0.00025 \) and \( c = 1.07 \)

Female survival after widowhood:

Makeham’s law, \( A = 0.0001, B = 0.00023 \) and \( c = 1.075 \)
Problem 3

Solution (a)

\[ i = 0.05, \quad A = 0.0001, \quad B = 0.0004, \quad c = 1.075 \]

\[ EPV = 100000 \dot{a}_{65}^{(12)} \]

Eqn. 5.18 says

\[ \dot{a}_{65}^{(12)} = \sum_{k=0}^{\infty} \frac{1}{12} \left( \frac{1}{1 + 0.05} \right)^{\frac{k}{12}} \frac{k}{12} p_{65} \]

Example 3.13 says for Makeham’s law:

\[ t \frac{d}{dt} p_x = \exp \left( -A t - \frac{B c^x (c^x - 1)}{\log(c)} \right) \]

Here:

\[ \frac{k}{12} p_{65} = \exp \left( -\frac{0.0001 k}{12} - \frac{0.0004 (1.075)^{65} (1.075^{\frac{k}{12}} - 1)}{\log(1.075)} \right) \]

\[ \dot{a}_{65}^{(12)} = 8.02693 \]

\[ EPV = 802693 \]
Notes about solution:

- $\ddot{a}_{x}^{(12)}$ - The expected present value of an annuity of amount 1 per year, payable in advance 12 times per year over the lifetime of $(x)$

- Eqn 5.19 states that

$$\ddot{a}_{65}^{(12)} = \sum_{k=0}^{\infty} \frac{1}{12} \frac{v^{k/12}}{p_{65}^{12}}$$ (6)

which weights the size of the benefit by the probability of surviving until a certain age, discounted by the time value of money

- Makeham’s Law: Models the force of mortality as the power law

$$\mu_{x} = A + Bc^{x}. \quad (7)$$

Then,

$$t p_{x} = \exp \left( - \int_{0}^{t} \mu_{x+s} \, ds \right). \quad (8)$$