Overview

1. Introduction

2. Multiple State Models

3. Problem Solving
Who Am I

My name: Thomas Merkh
My role: The TA of Math 178B

My background:
- Undergraduate degrees in Physics and Mathematics from RPI (NY State)
- This is my 4th year at UCLA, working on Mathematics PhD
- I research artificial intelligence, and have done research in nanotechnology, mathematical modeling for plasma physics, machine learning, and more.
- I have never taught an actuarial mathematics class, though I have been reading the book :)}
Important Information

- Office Hours: Every other week during this discussion section. Otherwise, Wednesday afternoons (times may vary week to week)

- Email: For this class, I will only be answering emails at tmerkh55@gmail.com (my previous spam inbox)

- Bi-weekly quizzes will be held Tuesdays and there will be a timeframe to submit to Gradescope. To get the true effect, we recommend taking the quiz over a 45 minute window.

- Every other week, there will be problem solving sessions on the topics that may appear on the following week’s quiz.
Course Content and Pre-requisites

Pre-requisites: Math 178A, or an equivalent class. This class will cover material on:

- Multiple State Models (Today)
- Multiple Decrement Models
- Profit Testing, Benefit Valuation
- Basic Distributions and probability theory
- Continuous Actuarial Models, and much more
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Questions?

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Essential aspects:

- Multiple state models are known as Markov processes with discrete states in continuous time.
- Generally, we have a set of states \( \{0, 1, \ldots, n\} \) with transition probabilities between them.
- The random variable \( Y(t) \) takes a value in \( \{0, \ldots, n\} \), and \( Y(t) = i \) means that an individual is in state \( i \) at age \( t \).

Assumptions:

- The next state doesn’t depend on information before current state (Markov Property).
- The is only one transition per “unit of time”.
1. Introduction

2. Multiple State Models

3. Problem Solving
Problem 1(a)

Suppose that Los Angeles has a weather state (sunny or rainy) described by the transition matrix:

\[ P = \begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix} \]  \hspace{1cm} (1)

The entry \( P_{11} = 0.9 \) indicates that a sunny day will be followed by another sunny day 90% of the time.

**Problem**

Today is sunny. What is the probability that it will be sunny the day after the day after tomorrow? Find this probability by listing all possible outcomes.
Problem 1(a)

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Solution

Let $S$ and $R$ be the events that it is sunny and rainy, respectively. Let $SSRS$ represent the event that it is rainy, sunny, and then sunny. Then the probability that the day after the day after tomorrow is sunny, call this $E$, given that today is sunny is

$$\Pr(S) = \Pr(SSSS | S) + \Pr(SSRS | S) + \Pr(SSRS | S) + \Pr(SRRS | S)$$

$$= 0.3 + 0.1(0.5)(0.9) + 0.1(0.5)(0.5) = 0.844.$$
Problem 1(a)

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$$

$$
= 0.9^3 + 0.1(0.5)(0.9) + 0.9(0.1)(0.5) + 0.1(0.5)(0.5) = 0.844.
$$
Suppose that today’s (sunny) state is represented as the probability vector

\[ x^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]

Similarly, the probability of tomorrow’s weather can be denoted by \( x^{(1)} \).

**Problem**

*Explicitly using the matrix \( P \), what is \( x^{(1)} \)?
Problem 1(b)

Problem

Explicitly using the matrix $P$, what is $x^{(1)}$?
Problem 1(b)

**Problem**

*Explicitly using the matrix $P$, what is $x^{(1)}$?*

**Solution**

$$x^{(1)} = x^{(0)} P = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.1 \end{pmatrix}.$$ (2)
Explicitly using the matrix $P$, what is $x^{(n)}$? Compare $x^{(3)}$ to the answer of part (a).
Problem 1(c)

Problem

Explicitly using the matrix $P$, what is $x^{(n)}$? Compare $x^{(3)}$ to the answer of part (a).

Solution

\[ x^{(2)} = x^{(1)} P = x^{(0)} P^2, \]

and following this pattern,

\[ x^{(n)} = x^{(0)} P^n. \]

Since $P^3$ is

\[ P^3 = \begin{pmatrix} 0.844 & 0.156 \\ 0.78 & 0.22 \end{pmatrix}, \]

one has that $x^{(3)} = \begin{pmatrix} 0.844 \\ 0.156 \end{pmatrix}$. 

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March 31, 2020
Figure: A two-state Markov chain describing the transitions between the sunny ($S$) and rainy ($R$) states. The probabilities shown here can be summarized by the transition matrix $P$ from previous slides.
Problem 2(a)

Since $P$ never changes and all states are repeatedly visited, there exists a stationary distribution describing the probability of sunny/rainy weather in Los Angeles. This distribution is independent of $x^{(0)}$, and can be denoted by $x^{(\infty)} = \lim_{n \to \infty} x^{(n)}$.

Problem

What would $x^{(\infty)} P$ be equal to?
Problem 2(a)

Since $P$ never changes and all states are repeatedly visited, there exists a stationary distribution describing the probability of sunny/rainy weather in Los Angeles. This distribution is independent of $x^{(0)}$, and can be denoted by $x^{(\infty)} = \lim_{n \to \infty} x^{(n)}$.

Problem

What would $x^{(\infty)} P$ be equal to?

Solution

Since $x^{(\infty)}$ is a stationary or stable distribution, $P$ has no effect on it, 

$$x^{(\infty)} P = x^{(\infty)}.$$
Problem 2(b)

**Problem**

*Using the answer from part (a), find $x^{(\infty)}$.***
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**Solution**

**Solve**

$$x^{(\infty)} P = x^{(\infty)}$$

$$x^{(\infty)} (P - I) = 0$$

$$\begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} -0.1 & 0.1 \\ 0.5 & -0.5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 5/6 \\ 1/6 \end{pmatrix}.$$  

*Notice that $x^{(\infty)}$ is the eigenvector of $P$ with eigenvalue 1.*
Problem 2(c)

In the long run, what percentage of days would be rainy? Sunny?

Solution

From the previous part, one would expect $\frac{1}{6} \approx 16.67\%$ of the days to be rainy on average.
Problem 2(c)

Problem

*In the long run, what percentage of days would be rainy? Sunny?*

Solution

*From the previous part, one would expect $1/6 \approx 16.67\%$ of the days to be rainy on average.*
Problem 2(d)

Problem

If

\[ P = \begin{pmatrix} 0.9 & 0.1 \\ 0.8 & 0.2 \end{pmatrix}, \quad \text{(3)} \]

how would the answer to part (c) change?
Problem 2(d)

**Problem**

If

\[ P = \begin{pmatrix} 0.9 & 0.1 \\ 0.8 & 0.2 \end{pmatrix}, \]  

how would the answer to part (c) change?

**Solution**

*Using a computer, one could approximate \( x^{(\infty)} \) by computing*

\[ P^{100} = \begin{pmatrix} 0.88889 & 0.11111 \\ 0.88889 & 0.11111 \end{pmatrix} \]

in MATLAB. Then one can see that approximately 11% of the days will be rainy on average.