SOME APPLICATIONS OF EXTREME VALUE THEOREM

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**Theorem 0.1** (existence). If \( f \) is a continuous function on domain \( D \) and \( D \) is closed and bounded, then \( f \) takes global min and max on \( D \).

**Theorem 0.2** (location). If \( f \) is a function on domain \( D \) and \( f \) takes global min or max, then they either occur at critical points in the interior of \( D \) or occur on the boundary of \( D \).

Notice that Theorem 0.2 does not have any assumptions on \( f \) or \( D \) (but keep in mind that critical points include points at which \( f \) doesn’t have partial derivatives!).

Sometimes the domain \( D \) is not closed (i.e. it misses some boundary points) or is not bounded (i.e. it is not contained in any ball of finite radius). Can we still show existence of global extrema? Sometimes we can, by applying Theorem 0.1 in an indirect way! In the following \( f \) is always a continuous function on \( D \).

(1) If \( D \) is not closed, we define the closure of \( D \) to be \( D \) plus all its missing boundary points, and denote this set \( \overline{D} \) (some denote it \( \text{cl}(D) \)). This is a closed set and is bounded whenever \( D \) is bounded. Sometimes \( f \) can be extended to a continuous function on \( \overline{D} \) (this is not always possible! e.g. \( f(x) = \frac{1}{x} \) and \( D = (0,1) \)). In this case, we can first solve the global extrema problem on \( \overline{D} \) for the extended function, and then check whether the extrema in \( \overline{D} \) actually lie inside \( D \).

(2) Let \( B \) be a closed and bounded set contained in \( D \). Sometimes we can reduce the problem of finding global extrema on the full domain \( D \) to finding global extrema on the subset \( B \) (to which we can apply Theorem 0.1). Namely,

(a) If there exists a point \( P \) inside \( B \) such that \( f(x) > f(P) \) for every point \( x \) in \( D \) outside \( B \), then global min of \( f \) on \( D \) exists and occurs inside \( B \).
(b) If there exists a point \( P \) inside \( B \) such that \( f(x) < f(P) \) for every point \( x \) in \( D \) outside \( B \), then global max of \( f \) exists and occurs inside \( B \).
(3) We can verify the conditions in (2) by explicitly finding such a subset $B$. But all we need to know is that such a subset $B$ exists; what it actually is doesn’t matter very much. This leads to the following criteria:

Let $S$ denote the set of missing boundary points of $D$. (If $D$ is open, $S$ is the whole boundary of $D$. If $D$ is closed, $S$ contains no point.)

(a) Global min on $D$ exists if there exists a point $P$ inside $D$ such that $f(x) > f(P)$ whenever $x$ is in $D$ and

(i) $|x|$ is large enough, or

(ii) $x$ is close enough to $S$.

(b) Global max on $D$ exists if there exists a point $P$ inside $D$ such that $f(x) < f(P)$ whenever $x$ is in $D$ and

(i) $|x|$ is large enough, or

(ii) $x$ is close enough to $S$.

Criterion (3a) / (3b) corresponds to criterion (2a) / (2b): if condition in (3a) / (3b) holds, then condition in (2a) / (2b) holds for some $B$. For example, we can take $B$ to be the set of points in $D$ within some maximal distance to $P$ and at some minimal distance away from $S$.

If $D$ is closed, then the criteria simplify as

(a) Global min on $D$ exists if there exists a point $P$ inside $D$ such that $f(x) > f(P)$ whenever $x$ is in $D$ and $|x|$ is large enough.

(b) Global max on $D$ exists if there exists a point $P$ inside $D$ such that $f(x) < f(P)$ whenever $x$ is in $D$ and $|x|$ is large enough.

If $D$ is bounded, then the criteria simplify as

(a) Global min on $D$ exists if there exists a point $P$ inside $D$ such that $f(x) > f(P)$ whenever $x$ is in $D$ and $x$ is close enough to $S$.

(b) Global max on $D$ exists if there exists a point $P$ inside $D$ such that $f(x) < f(P)$ whenever $x$ is in $D$ and $x$ is close enough to $S$.

1. Examples

1. Let $D$ be a closed subset of $\mathbb{R}^3$ and $P$ a point in $\mathbb{R}^3$. Show that there exists a point in $D$ closest to $P$. 
**Proof:** Let $P = (a, b, c)$. Define function $f(x, y, z) = (x - a)^2 + (y - b)^2 + (z - c)^2$; this is the square of the distance from $(x, y, z)$ to $P$. We want to show $f$ takes global min on $D$.

Note that $f(x, y, z) \to \infty$ when $|(x, y, z)| \to \infty$. Pick a point $Q$ in $D$. Then $f(x, y, z) > f(Q)$ when $|(x, y, z)|$ is large enough, so by criterion (3a), $f$ takes global min on $D$.

2. Show global min and max exists for

$$f(x, y) = (4y^2 - x^2)e^{-x^2-y^2}$$

on the whole plane.

**Proof:** Note that $f(x, y) \to 0$ as $|(x, y)| \to \infty$. Note that $f(1, 0) < 0$ and $f(0, 1) > 0$.

Pick $P = (1, 0)$, then $f(x, y) > f(1, 0)$ whenever $|(x, y)|$ is large enough, so by criterion (3a), global min exists.

Pick $P = (0, 1)$, then $f(x, y) < f(0, 1)$ whenever $|(x, y)|$ is large enough, so by criterion (3b), global max exists.

3. Show global min and max exists for

$$f(x, y) = \left(1 - \frac{x^2 + y^2}{10}\right) \sin \left(\frac{1}{1 - x^2 - y^2}\right)$$

on $D = \{(x, y) : x^2 + y^2 < 1\}$.

**Proof:** The function $f$ cannot be extended continuously to $\overline{D}$. To see this, write $u = x^2 + y^2$, then $f(x, y) = g(u) = (1 - \frac{u}{10}) \sin \left(\frac{1}{1-u}\right)$, for $0 \leq u < 1$. As $u \to 1$, $1 - \frac{u}{10} \to \frac{9}{10}$ and $\sin \left(\frac{1}{1-u}\right)$ oscillates infinitely many times between 1 and $-1$.

But note that $1 - \frac{u}{10}$ decreases to $\frac{9}{10}$ as $u$ increases to 1. Thus, if we take two numbers $0 \leq u_1, u_2 < 1$ such that $\sin \left(\frac{1}{1-u_1}\right) = 1$ and $\sin \left(\frac{1}{1-u_2}\right) = -1$ (e.g. $u_1 = 1 - \frac{2}{3}$, $u_2 = 1 - \frac{2}{3}$), then $g(u_2) < g(u) < g(u_1)$ for all $u > u_1, u_2$. (Visually, the amplitude of oscillation is decreasing as we approach the boundary.) By criteria (3a) and (3b), this shows $g$ takes global min and max for $0 \leq u < 1$, so $f$ takes global min and max on $D$.

2. **Problems**

1. Find three positive numbers that sum to 150 with the largest possible product of the three. Is there a smallest possible product?
Solution: We wish to maximize/minimize \( xyz \) subject to the conditions \( x, y, z > 0, x + y + z = 150 \). Using \( z = 150 - x - y \), this is the same as finding global max/min of the function \( f(x, y) = xy(150 - x - y) \) on the domain \( D = \{(x, y) : x, y > 0, 150 - x - y > 0\} \).

The domain \( D \) is a triangle in the plane but does not include the three sides, so it is not closed. However, \( f \) can be simply extended continuously to \( \overline{D} = \{(x, y) : x, y \geq 0, 150 - x - y \geq 0\} \) (the same triangle including the three sides).

We then solve the optimization problem on \( \overline{D} \). After calculations we found one critical point \((50, 50)\) in the interior of \( D \), and \( f(50, 50) = 125000 \). On the boundary (the three sides), \( f = 0 \).

Thus, global max on \( \overline{D} \) is 125000 and occur at \((50, 50)\), and the global min on \( \overline{D} \) is 0 and occur at every point on the boundary.

Restricting back to \( D \), global max on \( D \) is 125000 and occur at \((50, 50)\), and global min doesn’t exist on \( D \).

2. Show that the rectangular box, including the top and the bottom, with fixed volume \( V \) and smallest possible surface area is a cube. Is there a largest possible surface area?

Solution: Let \( x, y, z \) be the side lengths of the box. The surface area is \( 2xy + 2yz + 2xz \) and the volume is \( xyz \). Using that volume is equal to constant \( V \), we get \( z = \frac{V}{xy} \), and the surface area is \( f(x, y) = 2xy + \frac{2V}{x} + \frac{2V}{y} \). We wish to minimize/maximize \( f \) on the domain \( D = \{(x, y) : x > 0, y > 0\} \) (the interior of the first quadrant).

The function \( f \) cannot be extended continuously to \( \overline{D} \). In fact, we can see that \( f(x, y) \to \infty \) as \( |(x, y)| \to \infty \) or \( (x, y) \to S \), where \( S \) is the boundary of \( D \), which consists of the positive \( x \)-axis and the positive \( y \)-axis. This shows that global max on \( D \) doesn’t exist, because \( f \) can get arbitrarily large.

Moreover, pick \( P \) some point in \( D \), then this also shows \( f(x, y) > f(P) \) whenever \((x, y)\) is in \( D \) and \(|(x, y)| \) is large enough or \((x, y)\) is close enough to \( S \), so by criterion (3a), global min on \( D \) exists.

It remains to compute critical points of \( f \) in \( D \). (We don’t need to worry about the boundary because \( D \) is open, i.e. contains no boundary point). After calculations we found one critical point \((x, y) = (V^{1/3}, V^{1/3})\), so this is the global min. In this case, \( z = \frac{V}{xy} = V^{1/3} \), so the box is a cube with side length \( V^{1/3} \).