SUMMARY OF THE PAPER "A QUANTITATIVE HYDRODYNAMIC LIMIT OF THE KAWASAKI DYNAMICS"

TIANQI WU

This work is concerned with a *dynamical* version of the law of large numbers, called the *hydrodynamic limit*. This type of results say that under a suitable time-space rescaling, a deterministic dynamics emerges as the typical macroscopic behavior of a random evolution on a microscopic lattice as the system size goes to infinity. The particular microscopic evolution we study is the Kawasaki dynamics of 1-dimensional lattice systems of continuous, unbounded spins. It is a spin-exchange dynamics preserving the mean spin and, in the hydrodynamic limit, converges to a non-linear diffusion equation. On a qualitative level, this convergence had been established in [Fri87], [GPV88], and [LY93], but it is not apparent how to obtain quantitative rates of convergence from their methods.

The first step toward a quantitative theory was made in [GOVW09] by introducing the *two-scale approach*. In a nutshell, one chooses a *mesoscopic scale* between the microscopic and macroscopic scales and then defines a mesoscopic dynamics that is close to both the microscopic dynamics and macroscopic dynamics. The key insight behind the choice of the mesoscopic dynamics in [GOVW09] is a gradient flow interpretation of the microscopic and macroscopic dynamics, leading to the definition of the mesoscopic evolution in terms of an appropriately *coarse-grained* gradient flow structure. However, this perspective was not fully carried out in [GOVW09] due to technical reasons. In our work, we capitalize a lot more on the idea of choosing a mesoscopic evolution with the natural gradient flow structure, which leads to improved error estimates in the system size.

Let us now give more details. We consider a spin system consisting of N real-valued spins located on the 1-dimensional periodic lattice $\{1, \dots, N\}$. The associated *Hamiltonian* $H : \mathbb{R}^N \to \mathbb{R}$ for the spin values only has single-site potentials $\psi : \mathbb{R} \to \mathbb{R}$ and no interaction term:

$$H(x_1, x_2, \dots, x_n) := \sum_{i=1}^{N} \psi(x_i).$$

Department of Mathematics, UCLA

E-mail address: timwu@math.ucla.edu.

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The evolution of the spin values is governed via a coupled system of SDEs, called the *Kawasaki dynamics*:

$$dX_t = -A\nabla H(X_t)dt + \sqrt{2A}dB_t,$$

where B_t denotes a standard N-dimensional Brownian motion and $-A$ denotes the (centered) second-order difference operator for the periodic rescaled lattice $\left\{\frac{1}{N}, \dots, 1\right\}$. The presence of the matrix A means a site can only change its spin by distributing the difference to its neighbors. Consequently, the mean spin is conserved and we may restrict the dynamics to the hyperplane $X_N := \{x \in \mathbb{R}^N : \sum_{i=1}^N x_i = 0\}$.

As mentioned before, it is known that in the hydrodynamic limit $N \to \infty$, the Kawasaki dynamics converges to a nonlinear PDE on the 1-dimensional torus $\mathbb{T} = [0, 1]$

$$\partial_t \zeta = \partial_\theta^2 \varphi'(\zeta),$$

where the function $\varphi : \mathbb{R} \to \mathbb{R}$ is the *Cramér transform* of the single-site potential ψ, i.e.

$$\varphi(m) = \sup_{\sigma \in \mathbb{R}} \left(\sigma m - \log \int_{\mathbb{R}} \exp\left(\sigma z - \psi(z)\right) dz \right).$$

Compared with the microscopic dynamics,

- the 1-dimensional periodic lattice is rescaled and embedded into the 1-dimensional torus;
- the Euclidean space X_N is rescaled and embedded into the function space $L^2(\mathbb{T})$, as the subspace of piecewise constant functions;
- the second-order difference operator $-A$ is replaced by the second derivative ∂_θ^2;
- the microscopic Hamiltonian H is replaced by the *macroscopic free energy* $\mathcal{H}(\zeta) := \int_0^1 \varphi(\zeta) d\theta$, with ∇H and φ' as their gradient mappings in their respective spaces;
- and the noise term $\sqrt{2A}dB_t$ has disappeared.

Our aim is to make this statement of convergence *quantitative*, providing a good estimate on the speed of convergence, which is needed in applications. Towards this end we adapt the *two-scale approach* introduced in [GOVW09]. The basic idea is to choose a suitable *mesoscopic scale* between the microscopic and macroscopic scales on which we can define a mesoscopic dynamics η_t that serves as a *coarse-grained* version of both the microscopic dynamics X_t and macroscopic dynamics ζ_t:

- Relative to the microscopic scale of X_N, the mesoscopic observables should become more and more coarse-grained, which helps remove the random fluctuations of the microscopic dynamics by the law of large numbers.

- Relative to the macroscopic scale of $L^2(\mathbb{T})$, the mesoscopic observables should become more and more fine-grained, so that the coarse-grained dynamics approximate the full dynamics on the macroscopic scale.

After a suitable space Y of mesoscopic observables is chosen and embedded in the function space $L^2(\mathbb{T})$, the mesoscopic dynamics η_t will be given in the form of a (deterministic) high-dimensional ODE

$$\frac{d}{dt}\eta_t = -\bar{A}\nabla_{L^2}\bar{H}(\eta_t),$$

where $\bar{H}$ and $\bar{A}$ will be *coarse-grained* versions of H and A for the mesoscopic space Y.

The key feature of this set-up is that all three dynamics can now be viewed as *gradient flows*, i.e. the evolution of each dynamics reduces some kind of energy in the fastest possible way via some dissipation mechanism (see e.g. [JKO98, Ott01, AGS05, San16] for more details, examples, and further references).

- For the macroscopic dynamics ζ_t, the energy functional is $\mathcal{H}$ and the dissipation mechanism is the H^{-1} metric.
- For the mesoscopic dynamics η_t, the energy functional is $\bar{H}$ and the dissipation mechanism is the Euclidean metric on Y given by the inner product $\langle \cdot, \bar{A}^{-1}\cdot\rangle_{L^2}$.
- For the microscopic dynamics, the gradient flow structure appears on the level of the law ρ_t of the process X_t, which evolves according to the Fokker-Planck equation

$$\partial_t \rho_t = \nabla \cdot (\rho_t A \nabla H + A\nabla \rho_t).$$

The associated energy functional is then given by the *relative entropy* of ρ_t with respect to the *Gibbs equilibrium measure* $\mu(dx) := \frac{1}{Z}\exp(-H(x))dx$,

$$\mathrm{Ent}(\rho_t|\mu) = \int H(x)\rho_t(dx) + \int \log \rho_t(x)\rho_t(dx).$$

The dissipation mechanism is the *Wasserstein distance* on the space of probability measures over the Euclidean space X_N with the Euclidean metric given by the inner product $x\cdot A^{-1}y$,

$$W_2(\nu,\nu') := \min_{\gamma\in\Pi(\nu,\nu')}\left(\int_{X_N\times X_N}|x-x'|^2_{A^{-1}}\,d\gamma\right)^{\frac{1}{2}},$$

where $\Pi(\nu,\nu')$ is the set of all coupling of measures ν and ν' and $|\cdot|_{A^{-1}}$ denotes the Euclidean norm induced by the inner product $x\cdot A^{-1}y$.

dynamics	energy functional	dissipation mechanism
$\partial_t \zeta = \partial_\theta^2 \varphi'(\zeta)$	macroscopic free energy $\mathcal{H}(\zeta) := \int_0^1 \varphi(\zeta) d\theta$	H^{-1} metric
$\frac{d}{dt}\eta_t = -\overline{A}\nabla\overline{H}(\eta_t)$	coarse-grained Hamiltonian $\overline{H}$	$\overline{A}^{-1}$ metric
$dX_t = -A\nabla H(X_t)dt + \sqrt{2A}dB_t$ Fokker-Planck equation: $\partial_t \rho_t = \nabla \cdot (\rho_t A \nabla H + A\nabla \rho_t)$ where ρ_t is the law of X_t	microscopic free energy $\int H(x)\rho_t(dx) + \int \log \rho_t(x)\rho_t(dx)$ $:= \mathrm{Ent}(\rho_t \vert \mu)$ i.e. the *relative entropy* of the law ρ_t w.r.t. the *Gibbs measure* $\mu(dx) := \frac{1}{Z}\exp(-H(x))dx$.	Wasserstein distance of probability measures over the space X_N equipped with A^{-1} metric

It is known that the convergence of gradient flows may be deduced from the Γ-convergence of the associated energy functionals together with the convergence of the dissipation mechanisms in the proper sense (see e.g. [SS04, Ser11, Mie16]). Consequently, the gradient flow interpretation provides insights for the convergence of these dynamics. In terms of the energy functionals:

- The convergence of the coarse-grained Hamiltonian $\bar{H}$ to the macroscopic free energy $\mathcal{H}$ is a version of the *local Cramér theorem*. This input from equilibrium statistical mechanics also has the important consequence that $\bar{H}$ gains convexity from coarse-graining.
- The relative entropy $\mathrm{Ent}(\rho_t|\mu)$ can be interpreted as a *free energy* associated to the ensemble ρ_t, consisting of an energy part $\int H(x)\rho_t(dx)$ coming from the Hamiltonian H and an entropy part $\int \log \rho_t(x)\rho_t(dx)$ coming from the noise B_t.
  - Under coarse-graining onto mesoscopic profiles $y \in Y$, the entropy part vanishes due to the law of large numbers.
  - On the other hand, as the mesoscopic scale becomes more fine-grained, the energy part converges to the coarse-grained Hamiltonian $\bar{H}$ because the Kawasaki dynamics equilibrates faster on smaller spatial scales due to its spin-exchange mechanism. In the rigorous analysis this fact will be quantified with the help of a *log-Sobolev inequality* for the conditional Gibbs measures $\mu(dx|y)$ that is based on the convexity of $\bar{H}$.

In terms of the dissipation mechanisms:

- The operator $-\bar{A}$ is a coarse-grained version of the second-order difference operator $-A$ and therefore should converge to the second derivative ∂_θ^2.
- The dissipation mechanism on the microscopic level has two layers, an outer Wasserstein distance associated with the transportation of probability measures and an inner Euclidean

metric associated with the spin-exchange mechanism mediated by the matrix A. Under coarse-graining, the outer metric on the space of probability measures becomes degenerate as randomness vanishes, leaving only the inner metric on X_N associated with the A^{-1} inner product, which should then converge to the Euclidean metric on Y associated with the $\bar{A}^{-1}$ inner product.

To fully take advantage of the gradient flow interpretation, the operation of taking second-order difference should be compatible with coarse-graining, which means the mesoscopic observables need to be sufficiently smooth. In [GOVW09], the mesoscopic observables were chosen to be *piecewise constant functions*. The lack of regularity then forces one to use an unnatural definition of the coarse-grained operator $\bar{A}$ in the mesoscopic dynamics, leading to sub-optimal error estimates when that approach is taken to completion. In our work, we instead choose the mesoscopic observables to be *splines*, i.e. piecewise polynomials that are smoothly joined together. The extra regularity then allows a more natural definition of the mesoscopic dynamics in terms of its gradient flow structure, leading to overall better error estimates. On the other hand, the smoothness constraints of the splines also make them non-local functions, which makes deducing the main ingredients in the two-scale approach more subtle. As a workaround, we need to introduce another level of mesoscopic observables that are *piecewise polynomials*, on which we can perform the analysis more easily and then transfer the properties established there back to the original level of splines.

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