Partial difference equations over compact Abelian groups

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**SETTING**

\[ Z \quad \text{a compact (metrizable) Abelian group} \]

\[ U_1, \ldots, U_k \quad \text{closed subgroups of } Z \]

\[ \mathcal{F}(Z) \quad \{\text{measurable functions } Z \rightarrow T = \mathbb{R}/\mathbb{Z}\} \]

(\text{up to equality a.e.})

\[
\int_{Z} \cdot \, d\mu \quad \text{integral w.r.t. Haar probability measure}
\]

If \( w \in Z, f \in \mathcal{F}(Z) \) then \( d_wf(z) := f(z - w) - f(z) \)

— discrete analog of directional derivative.

If \( f : Z \rightarrow \mathbb{C} \), then \( \nabla_wf(z) := f(z - w)f(z) \)

— multiplicative variant
PARTIAL DIFFERENCE EQUATION

Describe those $f \in \mathcal{F}(\mathbb{Z})$ for which

$$d_{u_1} \cdots d_{u_k} f(z) \equiv 0 \quad \forall u_1 \in U_1, \ldots, u_k \in U_k.$$ 

ZERO-SUM PROBLEM

Describe those $f_1, \ldots, f_k \in \mathcal{F}(\mathbb{Z})$ s.t.

$$d_{u_i} f_i(z) \equiv 0 \quad \forall u_i \in U_i, \; i = 1, 2, \ldots, k$$

and

$$f_1(z) + f_2(z) + \cdots + f_k(z) \equiv 0.$$ 

Will focus on PD$^{ce}$Es in this talk.
MOTIVATION: SZEMERÉDI’S THEOREM

A \( k \)-AP in \( \mathbb{Z} \) is a set of the form

\[
\{z, z + r, \ldots, z + (k - 1)r \}
\]

for some \( z, r \in \mathbb{Z} \).

It is **non-degenerate** if all \( k \) entries are distinct.

**Theorem** (Szemerédi ’75).

\[
\forall \delta > 0 \ \forall k \geq 1 \ \exists N_0 = N_0(\delta, k) \geq 1 \ such \ that \ if \ N \geq N_0 \ and \ E \subseteq \mathbb{Z}_N \\ with \ |E| = \delta N, \ then
\]

\[
E \supseteq \text{some non-degenerate } k \text{-AP}.
\]
MULTI-DIMENSIONAL ANALOG

Fix $F = \{v_1, \ldots, v_k\} \subset \mathbb{Z}^d$, all $v_i$ distinct.

If $z \in \mathbb{Z}^d_N$ and $r \in \mathbb{Z}_N$, let

$$z + r \cdot F = \{z + rv_1, \ldots, z + rv_k\} \mod N.$$

Call this an $F$-constellation. It is non-degenerate if $|z + r \cdot F| = k$.

**Theorem** (Furstenberg and Katznelson).

$\forall \delta > 0 \; \forall d, k \geq 1 \; \exists N_0 = N_0(\delta, k, d) \geq 1$ such that if $N \geq N_0$ and $E \subseteq \mathbb{Z}^d_N$ with $|E| = \delta N^d$ then

$$E \supseteq \text{some non-degenerate } F\text{-constn.}$$

Szemerédi’s Theorem is special case

$$F = \{0, 1, \ldots, k - 1\} \subset \mathbb{Z}.$$
Many proofs now known, using graph theory (Szemerédi), ergodic theory (Furstenberg), hypergraph theory (Nagle-Rödl-Schacht, Gowers, Tao) or harmonic analysis/additive combinatorics (Roth, Gowers).

Roth’s and Gowers’ harmonic-analysis proof gives much the best bound on $N_0(\delta, k)$.

Some proofs generalize to higher dimensions. Roth-Gowers approach has not been generalized, and the known dependence of $N_0(\delta, k, d)$ is generally much worse when $d \geq 2$. 
SKETCH OF ROTH-GOWERS APPROACH

Will formulate this for multi-dimensional theorem as far as possible.

Important idea: estimate the fraction of all $F$-constellations that are contained in $E$. Let $Z = \mathbb{Z}_N^d$.

Suppose $\phi_1, \ldots, \phi_k : Z \rightarrow \mathbb{C}$, and set

$$\Lambda(\phi_1, \ldots, \phi_k) := \int_{\mathbb{Z}_N} \int_{\mathbb{Z}_N^d} \phi_1(z + r v_1)\phi_2(z + r v_2) \cdots \phi_k(z + r v_k) \, dz \, dr.$$

Interpretation: if $E \subseteq Z$ and $\phi_1 = \ldots = \phi_k = 1_E$, then

$$\Lambda(1_E, \ldots, 1_E) = \mathbb{P}(\text{random } F\text{-constn. lies in } E).$$
Idea is to show that if $|E| = \delta N$ then

$$\Lambda(1_E, \ldots, 1_E) \geq \text{const}(\delta, k) > 0.$$ 

Since

$$\mathbb{P}\left(\text{randomly-chosen } F\text{-constn. is degenerate}\right) \longrightarrow 0$$

as $N \longrightarrow \infty$, this proves the theorem.
Key tool for estimating $\Lambda$: **directional Gowers uniformity norms**.

Can formulate these for any compact $U_1, \ldots, U_k \subseteq Z$:

If $\phi : Z \to \mathbb{C}$, then

$$
\|\phi\|_{U(U_1, \ldots, U_k)} := \left( \int_{U_1} \int_{U_2} \cdots \int_{U_k} \int_{Z} \nabla u_1 \cdots \nabla u_k \phi(z) \, dz \, du_k \cdots du_1 \right)^{2^{-k}}.
$$
For example,

\[ \| \phi \|_4^{U(U_1, U_2)} = \int_{U_1} \int_{U_2} \int_{Z} \phi(z - u_1 - u_2) \phi(z - u_1) \phi(z - u_2) \phi(z) \, dz \, du_1 \, du_2. \]
**Theorem** (Gowers-Cauchy-Schwartz inequality).

\[ |\Lambda(\phi_1, \ldots, \phi_k)| \leq \|\phi_1\|_U(V_{12}, \ldots, V_{1k}) \cdot \|\phi_2\|_\infty \cdots \|\phi_k\|_\infty, \]

where \( V_{ij} := \mathbb{Z}_N \cdot (v_i - v_j) \leq \mathbb{Z}_N^d \) for \( 1 \leq i, j \leq k \).

Proved by repeated use of Cauchy-Schwartz inequality.

**Corollary.** If

\[ \|1_E - \delta\|_U(V_{12}, \ldots, V_{1k}) \approx 0, \]
\[ \|1_E - \delta\|_U(V_{21}, V_{23}, \ldots, V_{2k}) \approx 0, \]
\[ \vdots \]
\[ \|1_E - \delta\|_U(V_{k1}, \ldots, V_{k,k-1}) \approx 0, \]

then \( \Lambda(1_E, \ldots, 1_E) \approx \delta^k \), \( \implies \) many \( F \)-constellations in \( E \) if \( N \) is large.
Intuition: in this case, \( E \) ‘behaves like a random set’, and therefore ‘contains many \( F \)-constellations by chance’.

Idea for full proof: if

\[
\|1_E - \delta\|_{U(U_1,\ldots,U_{k-1})} \nRightarrow 0
\]

for one of the lists of subgroups above, then deduce some special ‘structure’ for \( E \), which implies positive probability of \( F \)-constellations for a different reason (not ‘by chance’). Proof is finished once have result for both ‘random’ and ‘structured’ \( E \).
In one-dimension, have: **Inverse Theorem for Gowers norms**.

In that case, $V_{ij} = \mathbb{Z}_N = \mathbb{Z}$ for every $i, j$.

Roughly: If $\|\phi\|_\infty \leq 1$ and $\|\phi\|_{U(Z,...,Z)} \geq \eta > 0$, then $\exists$ a ‘locally polynomial phase function’ $\psi : \mathbb{Z} \rightarrow S^1$ s.t.

$$\left| \int_{\mathbb{Z}} \phi(z)\psi(z) \, dz \right| \geq \text{const}(\eta) > 0.$$ 

Won’t define ‘locally polynomial phase functions’ here.
Related toy problem: suppose $|\phi(z)| \leq 1 \ \forall z$ but

$$\int_Z \cdots \int_Z \nabla u_1 \cdots \nabla u_{k-1} \phi(z) \ dz \ du_1 \cdots du_{k-1} = 1.$$ 

This is possible only if $\phi : Z \rightarrow S^1$ and

$$\nabla w_1 \cdots \nabla w_{k-1} \phi(z) \equiv 1.$$ 

Exercise: if $Z = \mathbb{Z}_N$, $N \gg k$ and $N$ prime, this occurs iff $\phi = \exp(2\pi i f/N)$ for some polynomial $f : \mathbb{Z}_N \rightarrow \mathbb{Z}_N$ of degree $\leq k - 2$.

There are more technical wrinkles for other $Z$, but still get some slightly-generalized notion of ‘polynomial of degree $\leq k - 2$’.
In higher dimensions no good inverse theorem known for directional Gowers norms.

Simplest toy case: describe those $\phi$ for which $|\phi(z)| \leq 1$ and

$$\|\phi\|_{U(U_1,\ldots,U_k)} = 1.$$ 

As before, this is equivalent to $\phi : \mathbb{Z} \rightarrow S^1$ and

$$\nabla_{u_1} \cdots \nabla_{u_k} \phi \equiv 1 \quad \forall u_1 \in U_1, \ldots, u_k \in U_k.$$ 

Writing $\phi = \exp(2\pi if)$ with $f : \mathbb{Z} \rightarrow \mathbb{T}$, this is exactly the partial difference equation we started with.
EXAMPLES OF PD<sup>ce</sup>E Es AND SOLUTIONS

Let us write our partial difference equation as

\[ d^U_1 \cdots d^U_k f \equiv 0. \]

**Example 1** If \( U_1 = \ldots = U_k = \mathbb{Z} \), then this is the case of ‘polynomial phase functions’ mentioned before.

**Example 2** There are always some obvious solutions: can take

\[ f = f_1 + \ldots + f_k, \]

where \( d^U_i f_i = 0 \) (equivalently, \( f_i \) is lifted from \( \mathbb{Z}/U_i \rightarrow \mathbb{T} \)).
Definition. If \( f \) solves
\[
d^{U_1} \cdots d^{U_k} f \equiv 0,
\]
then let’s say \( f \) is **degenerate** if can decompose it as
\[
f = f_1 + \cdots + f_k,
\]
where
\[
d^{U_2} d^{U_3} \cdots d^{U_k} f_1 = d^{U_1} d^{U_3} \cdots d^{U_k} f_2 = \cdots = d^{U_1} \cdots d^{U_{k-1}} f_k = 0.
\]
– that is, each \( f_i \) satisfies a simpler (and stronger) equation.

One is interested in classifying non-degenerate solutions, modulo degenerate ones.
Example 3  On $Z = \mathbb{T}^3$, let $f(\theta_1, \theta_2, \theta_3) = [\{\theta_1\} + \{\theta_2\}] \cdot \theta_3$.

(For $\theta \in \mathbb{T}$, $\{\theta\}$ is its unique representative in $[0, 1)$, and $[\cdot] =$ integer part.)

Then

\[
f(\theta_1, \theta_2, \theta_3) - f(\theta_1, \theta_2, \theta_3 + \theta_4) + f(\theta_1, \theta_2 + \theta_3, \theta_4) - f(\theta_1 + \theta_2, \theta_3, \theta_4) + f(\theta_2, \theta_3, \theta_4) = 0
\]

Think of these as five different functions of $(\theta_1, \theta_2, \theta_3, \theta_4) \in \mathbb{T}^4$.

By differencing, we can annihilate the last four terms, leaving

\[d^{U_1}d^{U_2}d^{U_3}d^{U_4}f = 0\]

for suitable $U_1, \ldots, U_4 \leq \mathbb{T}^4$. 
\[
f(\theta_1, \theta_2, \theta_3) - f(\theta_1, \theta_2, \theta_3 + \theta_4) + f(\theta_1, \theta_2 + \theta_3, \theta_4)
- f(\theta_1 + \theta_2, \theta_3, \theta_4) + f(\theta_2, \theta_3, \theta_4) = 0
\]

Disclosure: this equation is not arbitrary. It is the equation for a 3-cocycle in group cohomology \(H^3_m(\mathbb{T}, \mathbb{T})\).

One can prove that if \(f\) were degenerate, then it would be a coboundary: i.e., represent the zero class in \(H^3_m(\mathbb{T}, \mathbb{T})\).

But \(H^3_m(\mathbb{T}, \mathbb{T}) \cong \mathbb{Z}\), and I chose \(f\) above to be a generator. Hence non-degenerate.
More generally, $H^p_m(\mathbb{T}, \mathbb{T}) \cong \mathbb{Z}$ for all odd $p$. For each odd $p$, choosing a generator gives a non-degenerate solution to a PDCE with $p + 1$ subgroups.

This leads to many new ‘cohomological’ examples.
Example 4  Let $Z = \mathbb{T}^2 \times \mathbb{T}^2$, and define $\sigma, c : Z \rightarrow \mathbb{T}$ by

$$\sigma(s, x) = \{s_1\}\{x_2\} - \lfloor\{x_2\} + \{s_2\}\rfloor\{x_1 + s_1\} \mod 1$$

and

$$c(s, t) = \{s_1\}\{t_2\} - \{t_1\}\{s_2\} \mod 1$$

Then

$$\sigma(s, x + t) - \sigma(s, x) = \sigma(t, x + s) - \sigma(t, x) + c(s, t).$$

Interpret this as a zero-sum problem on $\mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$.

As for Example 3, repeated differencing annihilates all but one term, leaving a new PD$^{ce}$E for the subgroups

$$U_1 = \{(s, t, x) | s = x + t = 0\}, \quad U_2 = \{(s, t, x) | s = x = 0\},$$

$$U_3 = \{(s, t, x) | x + s = t = 0\}, \quad U_4 = \{(s, t, x) | s = t = 0\}.$$
\[
\sigma(s, x + t) - \sigma(s, x) = \sigma(t, x + s) - \sigma(t, x) + c(s, t).
\]

The reveal:

Written multiplicatively (i.e., for maps \( Z \to S^1 \)), this equation becomes

\[
\frac{\sigma(s, x + t)}{\sigma(s, x)} = c(s, t) \frac{\sigma(t, x + s)}{\sigma(t, x)}.
\]

This is the Conze-Lesigne equation, important in the study of two-step nilrotations.

Now a more complicated argument shows that \( \sigma \) is a non-degenerate solution to the PD\( \text{ce} \)E. Also: not obtained from a cocycle in group cohomology.
SOME POSITIVE RESULTS

Assume $U_1 + \ldots + U_k = Z$. (Otherwise, just work on each coset of $U_1 + \ldots + U_k$ separately.)

Let $M \subseteq F(Z)$ be the subgroup of solutions to the PD$^{ce}$E associated to $U_1, \ldots, U_k$. It is globally invariant under rotations of $Z$.

Let $M_0 \subseteq M$ be the further subgroup of degenerate solutions. It is also globally $Z$-invariant.

Lastly, let $| \cdot | : \mathbb{T} \rightarrow [0, 1/2]$ be distance from 0 in $\mathbb{T}$. 
**Theorem** (Small solutions are always degenerate).

\[ \forall k \geq 1 \exists \eta > 0 \text{ such that } f \in M \text{ and } \int_Z |f| < \eta \implies f \in M_0. \]

**Corollary.** The quotient $M/M_0$ is a countable discrete group.
**Theorem** (Stability for approximate solutions).

∀\(k \geq 1\) ∀\(\varepsilon > 0\) ∃\(\delta > 0\) such that if

\[
f \in \mathcal{F}(Z) \quad \text{and} \quad \int_{U_1} \cdots \int_{U_k} \int_Z |d_{u_1} \cdots d_{u_k} f(z)| < \delta
\]

then

\[
\exists g \in M \quad \text{s.t.} \quad \int_Z |f - g| < \varepsilon.
\]
Lastly, suppose $Z = \mathbb{T}^d$. A function $f : \mathbb{T}^d \rightarrow \mathbb{T}$ is step-polynomial if $\exists$ a partition

$$Z = P_1 \cup \cdots \cup P_m$$

such that

1. each $P_i$ is defined by linear inequalities in $\{\theta_i\}$ for $(\theta_1, \ldots, \theta_d) \in \mathbb{T}^d$, and

2. each restriction $f|_{P_i}$ is some polynomial function of $(\{\theta_i\})_{i=1}^d$, evaluated mod 1.
**Theorem** (Basic solns are step-polynomial, mod $M_0$). If $f \in M$, then

$$f = f_1 + g$$

for some step-polynomial $f_1 \in M$ and some $g \in M_0$. Also, have bounds on ‘complexity’ of $f_1$ that depend only on ‘quantitative smoothness’ of $f$, not on choice of $Z$: this gives a nontrivial result even for $Z$ finite.