Math 246a, Complex Analysis

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1 Comments about possible modifications of the lecture

1.1

Do derivatives of monomials right after the definition of complex differentiability.

1.2

differentiability should be defined for arbitrary sets rather than open sets

1.3

Discuss basic integration theory: Riemann integral as a continuos map from $C(a,b)$ with uniform topology to $\mathbf{C}$, and fundamental theorem of calculus.

1.4

3 miracle appears in the middle of proof of second miracle. sort this out better.

1.5

Starlike sets instead of convex sets for the lemma of Goursat. This includes the slit plane for the log.

1.6

Conformal should not include derivative 0

2 A product on $\mathbf{R}^2$: Complex numbers

We assume familiarity with the field $\mathbf{R}$ of real numbers. We also assume familiarity with vector spaces over $\mathbf{R}$, in particular we consider the vector space $\mathbf{R}^2$ consisting of all pairs (x, y) of real numbers with addition

$$(x, y) + (x', y') = (x + x', y + y')$$

and scalar multiplication ($\alpha \in \mathbf{R}$)

$$\alpha(x, y) = (\alpha x, \alpha y)$$

Complex Miracle 1 *One can define a product $\mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^2$ which makes $\mathbf{R}^2$ a field.*

We call this a miracle, because $\mathbf{R}^2$ is the only finite dimensional real vector space (other than the trivial cases $\mathbf{R}^1$ and $\mathbf{R}^0$) on which one can define such a product. Indeed, up to isomorphisms there is only one such product on $\mathbf{R}^2$. We will prove these remarks later in the course. One can define a product on the vector space $\mathbf{R}^4$ which satisfies axioms 1,2,4,5 (Quaternions: Exercise). Such a structure is called a skew field. On the vector space of real $n \times n$ matrices one has the matrix product, which satisfies axioms 1,2,4. Such a structure is called an algebra.

We define a map $p : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^2$ by

$$p((x, y), (x, y)) = (xx' - yy', xy' + x'y) \tag{1}$$

and write $(x, y)(x', y')$ for $p((x, y), (x, y))$

We claim that this map, which we call product, makes the vector space $\mathbf{R}^2$ a field, by which we mean that the product satisfies the following five axioms. We write z for an element $(x, y) \in \mathbf{R}^2$.

Lemma 1 *The product defined in (1) satisfies*

1. *Associative Law: For all $z, z', z'' \in \mathbf{R}^2$ we have $(zz')z'' = z(z'z'')$.*
2. *Distributive Law: For all $z, z', z'' \in \mathbf{R}^2$ we have $z(z' + z'') = zz' + zz''$.*
3. *Commutative law: For all $z, z' \in \mathbf{R}^2$ we have $zz' = z'z$.*
4. *Neutral element: There is a unique $z_1 \in \mathbf{R}^2$ such that for all $z \in \mathbf{R}^2$ $zz_1 = z$.*
5. *Inverse element: For all $z \in \mathbf{R}^2$ there is a $z' \in \mathbf{R}^2$ such that $zz' = z_1$*

Observe that uniqueness of z_1 indeed follows from the other axioms. Namely, if if z_1 and $z_1{}'$ are two neutral elements, then

$$z_1 = z_1 z_1' = z_1{}' z_1 = z_1{}'$$

The only reason for us to formulate uniqueness of z_1 in the axioms was to have z_1 at hand to formulate the existence of an inverse element. In fact, by a similar argument, one can see that the inverse element of an element z is unique (Exercise).

Proof of Lemma 1

One can verify all axioms by direct calculation. E.g., the commutative law follows from

$$(x,y)(x',y') = (xx'-yy', xy'+x'y) = (x'x-y'y, x'y+y'x) = (x',y')(x,y) \tag{2}$$

We will follow a slightly more enlightening path to see the other axioms.

We can write the multiplication (1) in matrix form

$$(x,y)(x',y') = (x,y)\begin{pmatrix} x' & y' \\ -y' & x' \end{pmatrix} \tag{3}$$

where the product on the right hand side is the ordinary matrix product.

Thus it is natural to consider the set M of all real 2×2 real matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

This set is a 2 dimensional subspace of the set of all real 2×2 matrices and there is a bijective linear map

$$m : \mathbf{R}^2 \to M$$

given by

$$m(x,y) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

We observe that for all $(x,y),(x',y') \in \mathbf{R}^2$ we have

$$m(x,y)m(x',y') = m((x,y)(x',y')) \tag{4}$$

where the product on the left hand side is the one given by (1) and the product on the right hand side is the usual matrix product. Indeed, (4) follows from

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\begin{pmatrix} x' & y' \\ -y' & x' \end{pmatrix} = \begin{pmatrix} xx'-yy' & xy'+yx') \\ -xy'-yx' & xx'-yy' \end{pmatrix}$$

Now we can verufy the associative law for (1) from the known associativity of the matrix product:

$$(zz')z'' = (zm(z'))m(z'') = z(m(z')m(z'')) = zm(z'z'') = z(z'z'')$$

Likewise we can prove the distributuive law from the distributive law of the matrix product. The matrix product has the neutral element

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is equal to $m(1,0)$. Hence we easily see that $(1,0)$ is a neutral element for the product (1).

Observe that while in general the matrix product is not commutative, on the space M it is, as follows from the calculation (2).

It remains to observe that each element in $\mathbf{C}$ other than $(0,0)$ has a multiplicative inverse. Observe first that for the determinant of a matrix in M we have

$$\det \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = x^2 + y^2$$

This is zero only if $x = 0$ and $y = 0$. Thus every nonzero element in M is invertible as a matrix. It remains to show that the inverse of an element in M is again in M. However, we have by Cramer's rule

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1} = \frac{1}{x^2+y^2} \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

Thus the inverse of the element (x,y) is given by

$$\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2})\right)$$

This completes the proof of Lemma 1. □

We call $\mathbf{R}^2$ with this structure of multiplication the field of complex numbers $\mathbf{C}$.

If α is a real number, we have the identity

$$\alpha(x,y) = (x,y) \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} = (\alpha, 0)(x,y) \tag{5}$$

Thus it is natural to identify a real number α with the complex numbers $(\alpha, 0)$ via the linear imbedding map $\mathbf{R} \to \mathbf{R}^2$ given by

$$\alpha \to (\alpha, 0) \quad .$$

A calculation as in (5) says that this is a field homomorphism.

It is customary to write i for the vector $(0,1)$. Then one writes

$$(x,y) = (x,0) + (0,1)(y,0) = x + iy$$

which is a popular representation of complex numbers. Observe that indeed

$$i^2 = (0,1)(0,1) = (0,1) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (-1,0)$$

Vector addition and scalar multiplication in $\mathbf{R}^2$ have well known geometrical meanings. We shall investigate the geometrical meaning of complex multiplication.

To do this type of geometric considerations, we recall some facts from basic geometry. A basic notion is the Euclidean length of two vectors

$$|z| := \sqrt{\langle z, z\rangle} = \sqrt{x^2 + y^2}$$

This formula is the Pythagorean theorem.

Related to the Euclidean length is the Euclidean inner product, of two vectors $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ in $\mathbf{R}^2$:

$$\langle (x_1, y_1), (x_2, y_2)\rangle = x_1x_2 + y_1y_2 = \frac{1}{2}\left(|z_1 + z_2|^2 - |z_1|^2 - |z_2|^2\right) \tag{6}$$

By the Pythagorean theorem again, if z_1 and z_2 are orthogonal, then the right hand side of the previous display vanishes and we have

$$\langle z_1, z_2\rangle = 0 \tag{7}$$

In fact, if (7) holds, then z_1 and z_2 are orthogonal, which is a converse to the Pythagorean theorem.

A linear map which preserves length of vectors is called an orthogonal linear map. By (6) such a map also preserves inner products. Let A be the matrix of such a map, then

$$\langle x, y\rangle = \langle xA, yA\rangle = \left\langle xAA^T, y\right\rangle \tag{8}$$

Since the jk-entry of a matrix B is given by

$$\left\langle e_jAA^T, e_k\right\rangle$$

where e_j denotes the j the standard basis vector of $\mathbf{R}^2$, we can use (8) to identify AA^T as the identity matrix. Vice versa, (8) implies that if AA^T is the identity matrix, then the map given by A orthogonal.

Given a vector $0 \neq z \in \mathbf{R}^2$, there are two orthogonal maps mapping $(1, 0)$ to z, because $(0, 1)$ is mapped to one of the two orthogonal vectors of z. One of these maps is a rotation, the other a reflection. The first one is given by a matrix with positive determinant, the second one by a matrix with negative determinant.

Now let A be the matrix of a linear map which preserves orthogonality, i.e.

$$\langle z_1, z_2\rangle = 0 \Rightarrow \langle z_1A, z_2A\rangle = 0$$

Then we can conclude eas above that AA^T is diagonal. Moreover, all diagonal entries are equal, as one can see from

$$\left\langle (e_j + e_k)AA^T, (e_j - e_k)\right\rangle = 0$$

for $j \neq k$. Thus AA^T is a multiple of the identity matrix. The entries are nonnegative, because

$$0 \leq \langle e_jA, e_jA\rangle = \left\langle e_jAA^T, e_j\right\rangle$$

Vice versa, if $AA^T = \alpha I$ with $\alpha > 0$, then $A/\sqrt{\alpha}$ is orthogonal and hence A preserves orthogonality.

A linear map which preserves orthogonality and has positive determinant is called a conformal map. Since it is a compostion of a scalar matrix with a rotation. Vice verse, any angle preserving map is a composition of a scalar with an orthogonal map, which needs to be a rotation because if it was a reflection it would reverse angles.

We now discuss the geometry of complex multiplication.

If $z = (x, y)$ is a complex number, we call x the real part and y the imaginary part of z, and write

$$\Re(z) = x, \quad \Im(z) = y$$

Observe that $\Re$ and $\Im$ are linear maps from $\mathbf{R}^2$ to $\mathbf{R}$.

Observe that the length of $z = (x, y) \in \mathbf{C}$ is related to the determinant of $m(z)$ via

$$|(x,y)| = \sqrt{\det\begin{pmatrix} x & y \\ -y & x \end{pmatrix}}$$

The product rule for determinants gives immediately the formula

$$|z||z'| = |zz'| \tag{9}$$

The complex conjugate of a number $z = (x, y)$ is defined to be

$$\overline{z} = (x, -y)$$

Observe that on the matrix side complex conjugation becomes transposition

$$m(\overline{z}) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}^T = m(z)^T$$

The product rule for transposed matrices gives

$$\overline{z}\,\overline{z'} = \overline{z'z} = \overline{zz'}$$

where in the last identity we have used the commutative law for complex numbers. Moreover, we have

$$\overline{z} + \overline{z'} = \overline{z + z'}$$

Thus complex conjugation is a field automorphism. (In fact this is the only linear map of $\mathbf{R}^2$ which is a field automorphism, because such automorphisms have to map 1 to 1 and i to $\pm i$ because of $i^2 = -1$.)

Now we have the following important identity:

$$|z|^2 = z\overline{z} \tag{10}$$

To see this, observe that the matrix of $\overline{z}z$ is a symmetric matrix. However, the only symmetric matrices in M are the multiples of the identity matrix. Hence $z\overline{z}$ is real. Then (10) follows from (9).

The identity

$$(x', y')(x, y) = (x', y') \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

shows that multiplication by $z = (x, y)$, i.e. the map $(x', y') \to (x', y')(x, y)$ is a linear map of $\mathbf{R}^2$ to $\mathbf{R}^2$. We shall see momentarily that this map is conformal. This is obvious if $(x, y) = z = 0$. Assume $z \neq 0$ then $z = (z/|z|)|z|$. Multiplication with $|z|$ is multiplication with a scalar, whereas multiplication by $z' = z/|z|$ is an orthogonal transformation:

$$m(z')m(z')^T = m(z')m(\overline{z'}) = m(z'\overline{z'}) = m(|z'|^2) = m(1)$$

Since the determinant of z' is positive, multiplication by z' is indeed given by a rotation. Since $1z' = z'$, the rotation angle is given by the angle betwewen 1 and z'. This is the same angle as the angle between 1 and z.

Lemma 2 *Complex multiplication by z gives a conformal linear map of $\mathbf{R}^2$. Vice versa, any conformal linear map of $\mathbf{R}^2$ is given by multiplication by a complex number.*

Proof: The first part has already been proved. Vice versa, a conformal map of $\mathbf{R}^2$ is easily seen to be determined by the image z of $(1, 0)$, namely the length of z gives the scalar and the angle of z gives the rotation angle. Thus this conformal map is identical to multiplication by z. □

Likewise, we call an affine linear mapping conformal, if it preseves angles between straight lines. If a and b are complex numbers, then

$$z \to az + b$$

is a conformal affine linear mapping. Vice versa, as a corollary to the previous lemma, every affine linear mapping is given in this form.

To see the strength of this geometrical interpretation, consider the question of finding roots of the equation

$$z^n = a \tag{11}$$

whenever a is a complex number. By elementary geometry we can write a as

$$a = (r \cos \phi, r \sin \phi)$$

where r is the length of a and ϕ is the angle of a to the vector $(1, 0)$. By the intermediate value theorem the positive number r has a positive root $r^{\frac{1}{k}}$. Then we immediately see that for

$$k = 0, 1, \ldots, n - 1$$

the complex number number

$$z = \left(r^{\frac{1}{k}} \cos \left(\frac{\phi}{n} + \frac{2\pi k}{n} \right), r^{\frac{1}{k}} \sin \left(\frac{\phi}{n} + \frac{2\pi k}{n} \right) \right)$$

solves equation (11).

3 Complex differentiability (versus total differentiability

We recall the notion of total differentiability.

We consider mappings

$$f : \mathbf{R}^n \to \mathbf{R}^m$$

or, more generally, mappings

$$f : \Omega \to \mathbf{R}^m$$

where Ω is an open subset of $\mathbf{R}^n$.

Such a mapping f is given by

$$f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

It is said to be totally differentiable at a point $X_0 = (x_{0,1}, \ldots, x_{0,n}) \in \Omega$, if there exists a linear map $\mathbf{R}^n \to \mathbf{R}^m$,

$$X \to XA$$

represented by a $n \times m$ matrix A, such that we have for sufficiently small X:

$$f(X_0 + X) = f(X_0) + XA + |X| r(X_0 + X)$$

where $r : \Omega \to \mathbf{R}^m$ vanishes at X_0 and satisfies

(12) $$\lim_{X \to 0} r(X_0 + X) = 0 \quad ,$$

i.e., r is continuous at X_0. Condition (12) can be read that A is a good linear approximation to

$$f(X_0 + X) - f(X_0)$$

at the point X_0.

This map A is then unique, because if A' was a different such linear map with remainder term r', then we had

(13) $$X(A - A') + |X|(r(X_0 + X) - r'(X_0 + X)) = 0$$

Replacing X by tX for small $t \in \mathbf{R}, t > 0$ and dividing by t gives

$$X(A - A') + |X|(r(X_0 + tX) - r'(X_0 + tX)) = 0$$

Letting $t \to 0$, the second term on the left hand side tends to 0, hence

$$X(A - A') = 0$$

Since this holds for all sufficiently small $X \in \mathbf{R}^n$, we obtain $A = A'$.

In this sense, A is the best linear approximation to $f(X_0 + X) - f(X_0)$ at X_0.

If f is totally differentiable at X_0, then the linear map A above is called the total derivative of f at X_0.

We recall the following facts about total differentiability:

Proposition 1 *If $\Omega \subset \mathbf{R}^n$ and $f : \Omega \to \mathbf{R}^m$ has a total derivative at $X_0 \in \Omega$, then its partial derivatives exist and its total derivative is given by the matrix of partial derivatives*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Vice versa, if all partial derivatives of f exist in an open neighborhood of X_0 and are continuous there, then f has a total derivative at X_0.

We also recall the chain rule, which say that the total derivative of the composition of two maps is the composition of the derivatives of the two maps.

Proposition 2 *Let $f : \mathbf{R}^n \to \mathbf{R}^m$ and $g : \mathbf{R}^m \to \mathbf{R}^k$ (these maps really only need to be defined on appropriate open sets). Let $X \in \mathbf{R}^n$ and assume f is totally differentiable at X_0 with total derivative A and g is totally differentiable at $f(X_0)$ with total derivative B, then $g \circ f$ is totally differentiable at X_0 with total derivative AB.*

Proof: We have

$$f(X_0 + X) = f(X_0) + XA + |X|r(X_0 + X)$$

Set $Y_0 = f(X_0)$, then we have

$$g(Y_0 + Y) = g(Y_0) + YA + |Y|s(Y_0 + Y)$$

Set $Y = XA + |X|r(X_0 + X)$ and observe that $Y \to 0$ as $X \to 0$. Then we have

$$g(f(X_0 + X)) = g(f(X_0)) + XAB + [|X|r(X_0 + X)B + |Y|s(Y_0 + Y)]$$

Then the last term on the right hand side tends to 0 as $X \to 0$, hence AB is the total derivative of $g \circ f$ at X_0. $\square$

We now consider the special case of maps from $\mathbf{R}^2$ to $\mathbf{R}^2$, or rather form open subsets of $\mathbf{R}^2$ to $\mathbf{R}^2$. Given the complex structure on $\mathbf{R}^2$, it should be of interest whether the total derivative at a point is given as multiplication by a complex number. Thus we make the following definition:

Definition 1 *Let $\Omega \subset \mathbf{R}^2$ be an open subset. A function*

$$f : \Omega \to \mathbf{R}^2$$

$$f : (x, y) \to (u(x, y), v(x, y))$$

is said to be complex differentiable at a point $z_0 = (x_0, y_0)$, if it is totally differentiable and its total derivative at z_0 is given by multiplication by a complex number, i.e.,

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} \in M$$

Then there is a unique complex number which we call $f'(z_0)$ which satisfies

$$m(f'(z_0)) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Thus, if f is complex differentiable at a point $z_0 = (x_0, y_0)$, then its partial derivatives exist at the point z_0 and satisfy the *Cauchy-Riemann differential equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

at this point.

Let f be complex differentiable at z_0. Observe that we have

$$f(z_0 + z) = f(z_0) + zf'(z_0) + |z|r(z_0 + z)$$

and thus

$$\frac{|z|}{z}r(z_0 + z) = \frac{f(z_0 + z) - f(z_0)}{z} - f'(z)$$

Since

$$\lim_{z \to 0} r(z_0 + z) = 0$$

we have

$$\lim_{z \to 0} \frac{|z|}{z}r(z_0 + z) = 0$$

because the norm of both expressions inside the limits is the same. Hence

$$\lim_{z \to 0} \frac{f(z_0 + z) - f(z_0)}{z}$$

exists and is equal to $f'(z)$. Vice versa, one can see by reversing these arguments that if this limit exists, then f is complex differentiable at the point z_0 and this limit is the complex derivative. This explains the notion of complex differentiability. Observe that we have used the structure of complex multiplication to divide by z and thus write a difference quotient.

As for the derivative of one real variable we have the following rules:

Lemma 3 *If f and g are complex differentiable at z_0, then so are its sum and product, and the derivatives at z_0 are given by*

$$(f + g)' = f' + g'$$

$$(fg)' = f'g + fg'$$

If f is complex differentiable at z_0 and g is complex differentiable at $f(z_0)$, then the chain rule implies

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$$

We omit the proof.

A slightly stronger and eventually more important definition than that of complex differentiability is that of holomorphy.

Definition 2 *Let Ω be an open subset of $\mathbf{R}^2$. A map*

$$f : \Omega \to \mathbf{R}^2$$

is said to be holomorphic at a point $z \in \Omega$ if it is complex differentiable at every point in an open neighborhood of z. It is called holomorphic in Ω, if it is holomorphic at every point of Ω.

Next we shall consider maps from $\mathbf{R}$ to $\mathbf{R}^2$, or, more generally, maps from an open interval of $\mathbf{R}$ to $\mathbf{R}^2$. Such maps are called curves for obvious reasons: considering the parameter $t \in \mathbf{R}$ as time, such maps describe a point wandering along a curve

$$\gamma(t) = (x(t), y(t))$$

The total derivative at a point , if it exists, is called the tangent vector to γ at $z = \gamma(t)$. It is given by

$$\gamma'(t) = (x'(t), y'(t))$$

We can read both $\gamma(t)$ and the tangent vector $\gamma'(t)$ as a complex number.

Now let $z \in \Omega$ for some open domain Ω and

$$f : \Omega \to \mathbf{C}$$

be a map which is complex differentiable at the point z. By shrinking I if necessary we may assume $\gamma : I \to \Omega$. Then there is a curve

$$f \circ \gamma : I \to \mathbf{C}$$

given by

$$f \circ \gamma(t) = f(\gamma(t))$$

The tangent vector of $f \circ \gamma$ at $f(z)$ is given by the chain rule as

$$f'(z)\gamma'(t)$$

Observe that $f'(z)$ is a conformal map.

Hence, if $\tilde{\gamma} : I \to \Omega$ is a different curve through the point z, then conformality of $f'(z)$ implies that the angle between the tangent vectors of γ and $\tilde{\gamma}$ at z is the same as the angle between the tangent vectors of $f \circ \gamma$ and $f \circ \tilde{\gamma}$ at $f(z)$.

Differentiable maps from an open subset of $\mathbf{R}^2$ to $\mathbf{R}^2$ which preserve angles in the above sense are called conformal. We have seen that holomorphic maps are conformal. As we have already proved for linear maps, the converse also holds: conformal maps are holomorphic.

4 Integral formulas

The fundamental theorem of calculus says that for "nice" functions $f_{[a,b]\to\mathbf{R}}$ we have

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

For our purposes it is sufficient to interpret "nice" that the function f is differentiable in $[a,b]$ (where at the boundary points we have one-sided derivatives) and is cointinuous in $[a,b]$.

Clearly the fundamental theorem of calculus can be applied to the components of a map $f\circ\gamma$ where

$$\gamma : [a,b] \to \mathbf{R}^n$$

is a curve with continuous derivative (both components continuously differentiable in the above sense) and

$$f : \Omega \to \mathbf{R}^n$$

is a totally differentiable map so that the total derivative (all components of the matrix) is continuous and the open set Ω contains the image of γ.

In the case of $\mathbf{R}^2$ and holomorphic f we obtain the following lemma

Theorem 1 *Let $\gamma[a,b] \to \mathbf{R}^2$ be continuously differentiable, let $f : \Omega \to \mathbf{R}^2$ be holomorphic with continuous derivative, and assuem $\Omega \subset \mathbf{R}^2$ contains the image of γ. Then we have*

$$\int_a^b f'(\gamma(t))\gamma'(t)\,dt = f(\gamma(b)) - f(\gamma(a)) \tag{14}$$

Proof: By the chain rule we have

$$f'(\gamma(t))\gamma'(t) = \frac{d}{dt}(f\circ\gamma(t))$$

By assumptions the right hand side is continuous. Thus the theorem follows from the fundamental theorem of calculus. □

We stress that this theorem has nothing to do with particularities of complex differentiability: the analoguous statement holds in the setting of total differentiability.

As a special case we might have $\gamma(a) = \gamma(b)$, in which case the right hand side of (14) vanishes. While no interesting analogue of this exists in one dimension, in higher dimensions this might occur for nontrivial closed curves.

We observe that Formula (14) makes sense also when γ continuous but only piecewise continuously differentiable. Bu this we mean we can partition the domain $[a,b]$ of γ into sub intervals $[a_1,b_1],\ldots,[a_k,b_k]$ such that if γ_j is the restriction of γ to $[a_j,b_j]$ then $gamma_j$ is continuously differentiable (one -sided derivatives at the end points. If $\gamma_1,\ldots,\gamma_n$ are these pieces, then we define

$$\int_a^b f'(\gamma(t))\gamma'(t)\,dt = \sum_{j=1}^k \int_{a_j}^{b_j} f'(\gamma_j(t)){\gamma_j}'(t)\,dt$$

By *curve* we will form now on always mean a piecewise continuously differentiable curve as above. It will be called closed if $\gamma(a) = \gamma(b)$.

We call γ closed, if initial and terminal point coincide. As we have seen, for a complex differentiable function f whose derivative is continuous, we have

$$\int_\gamma f'(\gamma(t))\gamma'(t)\,dt = 0 \tag{15}$$

whenever the image of γ is inside the domain of f.

Complex Miracle 2 *Identity (15) also holds with f' replaced by f, provided f is holomorphic in a convex open set Ω which contains the curve γ. In fact, validity of (15) for all γ inside Ω is a characterization of holomorphy of f.*

We begin by establishing (15) with f replaced by f' for holomorphis functions inside a triangle. This will be as good as the general case, as we will see momentarily.

Lemma 4 (Goursat) *Let f be holomorphic in an open set Ω and let z_1, z_2 z_3 be three points in Ω such that their convex hull is also in Ω. Let*

$$\gamma_1(t) = z_2 t + z_1(1-t)$$

$$\gamma_2(t) = z_3 t + z_2(1-t)$$

$$\gamma_3(t) = z_1 t + z_3(1-t)$$

where the parameter t runs from 0 to 1 for each path. Then

$$\sum_{j=1}^{3} \int_{\gamma_j} f(\gamma(t))\gamma'(t)\,dt = 0$$

Proof: We make the preliminary observation that if

$$\gamma : [a,b] \to \mathbf{R}^2$$

is a path, then

$$\int_a^b f'(\gamma(t)\gamma'(t)\,dt = -\int_a^b f'(\gamma(b+a-t)\gamma'(b+a-t)\,dt \tag{16}$$

This is an elementary calculation.

We call the quantity defined in the lemma $I(z_1, z_2, z_3)$, and will consider analogu-ous quantities for other triples of points. Let v_3, v_1, v_2 be the midpoints of z_1z_2, z_2z_3, z_3z_1. Connecting these points partitions the original triangle into four congruent triangles which have half the size of the original one.

Writing each integral as a sum and using (16), a simple geometry observation shows that

$$I(z_1, z_2, z_3) = I(z_1, v_3, v_2) + I(v_3, z_2, v_1) + I(v_1, z_3, v_2) + I(v_1, v_3, v_2)$$

Let $z_1^{(1)}$ $z_2^{(1)}$, $z_3^{(1)}$ be the triangle on the right hand side for which $|I(z_1^{(1)}, z_2^{(1)}, z_3^{(1)})|$ is maximal. Then, by the triangle inequality,

$$|I(z_1, z_2, z_3)| \leq 4|I(z_1^{(1)}, z_2^{(1)}, z_3^{(1)})|$$

Definr $\mathrm{diam}(\mathrm{z}_1, \mathrm{z}_2, \mathrm{z}_3)$ to be the maximum of $|z_1 - z_2|$, $|z_2 - z_3|$, $|z_3 - z_1|$, then we have

$$\mathrm{diam}(z_1^{(1)}, z_2^{(1)}, z_3^{(1)}) = 2^{-1}\mathrm{diam}(z_1, z_2, z_3)$$

We iterate this construction by subdividing the triangle $z_1^{(1)} z_2^{(1)} z_3^{(1)}$ further, thus obtaining a sequence of nested triangles $z_1^{(n)} z_2^{(n)} z_3^{(n)}$. Since a nested sequence of compact sets has nonempty intersection, we find a point z_0 in the intersection of all these triangles.

We can write

$$f(z_0 + z) = f(z_0) + f'(z_0)z + |z|r(z_0 + z) \tag{17}$$

with $\lim_{z\to 0} r(z_0 + z) = 0$. Obviously the function

$$g(z) = f(z_0) + f'(z_0)z$$

is the derivative of

$$G(z) = f(z_0)z + \frac{1}{2}f'(z_0)z^2$$

Hence any integral of g over a closed curve is 0 (that is the fundamental theorem of calculus). By the definition of differentiability we find for each $\epsilon > 0$ a ball about z_0 such that $|r(z)| < \epsilon$ for z in this ball. Choose n large enough so that the triangle $z_1^{(n)} z_2^{(n)} z_3^{(n)}$ lies inside this ball. Then we have

$$|I(z_1^{(n)}, z_2^{(n)}, z_3^{(n)})| \leq$$

$$\sum_{j=1}^{3} \int_{\gamma_j^{(n)}} |z - z_0| r(\gamma_j^{(n)}(t)) \gamma_j^{(n)\prime}(t)|\, dt$$

(Here we have used (17) and neglected the linear term)

$$\leq C \sum_{j=1}^{3} 2^{-n} \epsilon \int_{\gamma_j^{(n)}} |\gamma_j^{(n)\prime}(t)|\, dt$$

Here we have used

$$\mathrm{diam}(z_1^{(1)}, z_2^{(1)}, z_3^{(1)}) = 2^{-n}\mathrm{diam}(z_1, z_2, z_3) = 2^{-n}C$$

Finally we have $|\gamma_1^{(n)\prime}| = |z_2^{(n)} - z_1^{(n)}|$ etc, so we obtain

$$|I(z_1^{(n)}, z_2^{(n)}, z_3^{(n)})| \leq C^2 2^{-2n}\epsilon$$

However,

$$|I(z_1, z_2, z_3)| \leq 4^n |I(z_1^{(n)}, z_2^{(n)}, z_3^{(n)})|$$

$$\leq C^2\epsilon$$

Since ϵ was arbitrary, we obtain

$$I(z_1, z_2, z_3) = 0$$

This had to be proved. □

We stress that the point where we have used complex differentiability is when we have observed that the linear map

$$z \to f'(z_0)z$$

is the derivative of a holomorphic function, so that we could use the fundamental theorem of calculus to argue this term away.

We also point out that in case the complex derivative of f is continuous, we can prove this theorem using Stokes' theorem.

Consider only the first component of the integral (15).

$$\int_a^b u(\gamma(t))x'(t) - v(\gamma(t))y'(t)\, dt$$

$$= \int_\Delta \left(\pm\frac{\partial u}{\partial y}(x,y) \mp \frac{\partial v}{\partial x}(x,y) \right) dxdy$$

Here the integrand on the right hand side is the curl of the vector field which is integrated on the left hand side (with a sign depending on the orientation of the curve γ) and Δ is the interior of the (say) triangle formed by γ. The Cauchy-Riemann differential equations say that the integrand on the right hand side vanishes. Similarly we obtain vanishing of the second component bythe other Cauchy Riemann differential equation.

Our proof of the Lemma of Goursat is nicer however, because it does not need any assumptions on the regularity of the derivative of f. We note an even stronger version:

Corollary 1 *The previous lemma continues to hold if f is continuous in Ω and complex differentiable at all points in the the convex hull of z_1,z_2, z_3 with one possible exception p*

Proof: Since f is continuous, it is bounded by a constant C on the triangle. Assume first p is one of z_1, z_2, z_3. Then we can run the previous algorithm. Observing that at each step only one triangel contains p and the integral of the other triangles is 0 by the lemma, we indeed obtain

$$|I(z_1, z_2, z_3)| \leq |I(z_1^{(n)}, z_2^{(n)}, z_3^{(n)})|$$

If the diameter of $I(z_1^{(n)} z_2^{(n)} z_3^{(n)}$ becomes less than ϵ, then

$$|I(z_1^{(n)}, z_2^{(n)}, z_3^{(n)})| \leq 3C\epsilon$$

Thus again

$$I(z_1, z_2, z_3) = 0$$

If p is none of z_1, z_2, z_3, then we can subdivide the triangle into 2 or 3 triangles, each of which has p as a corner and then apply the previous reasoning. □

Recall that a convex domain is one such that for any two points in the domain, the connecting line between the two points is also in the domain.

We have the following

Lemma 5 *Let Ω be an open convex domain in $\mathbf{R}^2$. Let $f : \Omega \to \mathbf{R}^2$ be a holomorphic function. Then there is a holomorphic function*

$$F : \Omega \to \mathbf{R}^2$$

such that $F' = f$

Proof:

Pick a point $z_0 \in \Omega$. For each $z_1 \in \Omega$ define

$$\gamma(t) = tz_1 + (1-t)z_0$$

where t runs from 0 to 1.

Then define

$$F(z_1) = \int_0^1 f(\gamma(t))\gamma'(t)\,dt$$

We claim F is holomorphic in Ω and satisfies $F' = f$.

Consider the difference quotient

$$\frac{F(z_2) - F(z_1)}{z_2 - z_1}$$

By definition of F and by the Lemma of Goursat we can write this as

$$\frac{1}{z_2 - z_1}\int_0^1 f(\gamma(t))\gamma'(t)\,dt = \int_0^1 f(\gamma(t))\,dt$$

where γ is defined as

$$\gamma(t) = tz_2 + (1-t)z_1$$

and we have used $\gamma'(t) = z_2 - z_1$.

Since f is holomorphic, we can write

$$f(z) = f(z_1) + (z - z_1)f'(z_1) + |z - z_1|r(z)$$

with a function $r(z)$ which is bounded in a small neighborhood of z_1. Thus we obtain

$$\int_0^1 f(\gamma(t))\,dt = f(z_1) + \int_0^1 ((z - z_1)f'(z_1) + |z - z_1|r(z))dt$$

The second summand on the right hand side we can estimate by

$$\left| \int_0^1 (z - z_1) f'(z_1) + |z - z_1| r(z) dt \right| \leq C(z_1) |z_2 - z_1|$$

for z_2 close to z_1. Therefore,

$$\lim_{z_2 \to z_1} \frac{F(z_2) - F(z_1)}{z_2 - z_1}$$

exists and is equal to $f(z_1)$. This proves the lemma □

Observe that we have also proved the following

Corollary 2 *Let Ω be a convex open subset of $\mathbf{R}^2$ and let $f : \Omega \to \mathbf{R}^n$ be a function that satisfies*

$$\int_\gamma f(\gamma(t))\gamma'(t)\, dt = 0 \tag{18}$$

where γ is any triangular curve as in the Lemma of Goursat. Then (18) is satisfied for all closed curves γ in Ω.

Proof: As in the proof of the previous lemma, f has a primitive F, thus the statement follows from the fundamental theorem of calculus. □

The circle of ideas around the Lemma of Goursat and the integrability criterion above has an analogue in severeal real variables, which we now discuss. The following discussion is purely formal: we shall always assume that all functions are sufficiently nice so that integrals and derivatives exist. We will develop the full theory with proofs in the next lecture.

Let Ω be an open convex subset of $\mathbf{R}^n$. A map $f : \Omega \to \mathbf{R}^n$ is called integrable, if there exists a function $F : \Omega \to \mathbf{R}$ such that

$$\frac{\partial F}{x_j} = f_j \tag{19}$$

for all $j = 1, \ldots, n$. Then the fundamental theorem of calculus (one direction) and an analoguous argument as above (converse direction, convexity is been used) say that f is integrable if and only if

$$\int_0^{2\pi} \sum_j f_j(\gamma(t))\gamma_j'(t)\, dt = 0 \tag{20}$$

for every closed curve $\gamma : [0, 2\pi] \to \mathbf{R}^n$

If there is a map $\Gamma : D \to \mathbf{R}^n$, where D is the closed unit disc in $\mathbf{R}^2$ such that

$$\gamma(\theta) = \Gamma(\cos(\theta), \sin(\theta))$$

(always exists if Ω is convex) then Stokes' theorem says that

$$\int_0^{2\pi} \sum_j f_j(\gamma(\theta))\gamma_j'(\theta)\, d\theta =$$

$$\int_0^1 \int_0^{2\pi} \sum_{j,k=1}^{n} \left(\frac{\partial f_j}{\partial x_k}(\Gamma(r,\theta)) - \frac{\partial f_k}{\partial x_j}(\Gamma(r,\theta)) \right) \frac{\partial \Gamma_k}{\partial r}(r,\theta) \frac{\partial \Gamma_j}{\partial \theta}(r,\theta)\, d\theta dr$$

Thus, if Ω is convex, so that every map γ can be extended to a map Γ, then Stokes theorem gives the sufficient "integrability condition"

$$\frac{\partial f_j}{\partial x_k} - \frac{\partial f_k}{\partial x_j} = 0 \tag{21}$$

for f to be integrable. This condition is also necessary, because if (19) holds then we have

$$\frac{\partial f_j}{\partial x_k} - \frac{\partial f_k}{\partial x_j} = \frac{\partial^2 F}{\partial x_k \partial x_j} - \frac{\partial F}{\partial x_j \partial x_k} = 0$$

We now specialize again to the case of $\mathbf{R}^2$ with its complex structure. Let Ω be an open convex domain in $\mathbf{R}^2$ and let $f : \Omega \to \mathbf{R}^2$,

$$f(x,y) = (u(x,y), v(x,y))$$

be a holomorphic function.

Consider only the first component of the integral

$$\int_a^b f'(\gamma(t))\gamma'(t)\, dt$$

It is given by

$$\int_a^b (u(\gamma(t))x'(t) - v(\gamma(t))y'(t))\, dt$$

Thus this integral is of the form (20). The corresponding integrability condition (21) is

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

This is one of the Cauchy Riemann equations. Likewise the second component of the above complex integral is given by

$$\int_a^b v(\gamma(t))x'(t) + u(\gamma(t))y'(t)\, dt$$

The corresponding integrability condition (21) is

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

which is the other Cauchy Riemann equation.

In summary, we can reformulate the second complex miracle as follows:

Complex Miracle 2 *The algebraic complex differentiability condition, namely the Cauchy Riemann differential equations, is the same as the algebraic complex integrability condition.*

An explanation for this miracle is the fact that complex polynomials are both differentiable and integrable, which suggests that there should be a close connection between differentiability and integrability conditions. After all, complex polynomials of small degree played an important role in the proof of the Lemma of Goursat.

5 The Cauchy integral formula

A striking application of the more general version of the Lemma of Goursat leads to the next miracle:

Complex Miracle 3 *The value of a complex function is determined by its values far away, in fact it is an average of values far away.*

We shall introduce some simplified notation. If $\gamma : [a,b] \to \mathbf{R}^2$ is a piecewise continuously differentiable curve, then we write

$$\int_\gamma f(\gamma(t))\gamma'(t)\,dt = \int_\gamma f(z)\,dz$$

A precise version of the third complex miracle is given by the following lemma:

Lemma 6 (Cauchy's integral formula) *Let Ω be a convex open domain. Let $f : \Omega \to \mathbf{R}^2$ be a holomorphic function. Then, for every $z_0 \in \Omega$ and every closed curve $\gamma \in \Omega$ whose image does not contain z_0 we have*

$$f(z_0)\int_\gamma \frac{1}{z-z_0}\,dz = \int_\gamma \frac{f(z)}{z-z_0}\,dz$$

Proof: Consider the function

$$g(z) = \frac{f(z)-f(z_0)}{z-z_0}$$

Observe that the function $z \to 1/z$ is differentiable away from 0:

$$\lim_{z_1 \to z_2} \frac{\frac{1}{z_1} - \frac{1}{z_2}}{z_1 - z_2} = -\lim_{z_1 \to z_2} \frac{1}{z_1 z_2} = -\frac{1}{{z_1}^2}$$

By the elementary rules for derivatives the function g is differentiable in $\Omega \setminus z_0$. At z_0 it is continuous by definition of differentiability. Thus we can apply the more general version of the lemma of Goursat to obtain

$$\int_\gamma g(z)\,dz = 0$$

where γ is a triangular curve, and then by the discussions of the second complex miracle for every closed curve γ. This implies Cauchy's integral formula. $\square$

Of course the interesting point of Cauchy's integral formula is that the integral

$$\mathrm{Ind}_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz$$

need not be 0 because the function $1/(z-z_0)$ does not satisfy the conditions of the general Lemma of Goursat: it is not continuous at the point z_0. We will discuss the number $\mathrm{Ind}_\gamma(z_0)$ in more detail later, for now we observe that for a small circle about z_0,

$$\gamma(t) = z_0 + (r\cos(t), r\sin(t)) \quad ,$$

we have

$$\gamma'(t) = (-r\sin(t), r\cos(t)) = i(\gamma(t) - z_0)$$

and thus

$$\mathrm{Ind}_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}\,dz$$

$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{i(z - z_0)}{z - z_0}\,dt = 1$$

In particular we see that the value of f at z_0 is determined by an integral over values of f outside an open neighborhood of z_0, which is what the third complex miracle says.

The third complex miracle is in stark contrast to the situation of real valued functions, even if they are infinitely often differentiable. Such functions still can take any values at a given point even if we specify their values outside a small neighborhood of the point. This can be seen from the following example $f : \mathbf{R} \to \mathbf{R}$

$$f(x) = \begin{cases} Ce^{-\frac{\epsilon^2}{(x^2-\epsilon^2)}} & |x| < \epsilon \\ 0 & |x| \geq \epsilon \end{cases}$$

which is infinitely often differentiable, takes the value C/e at 0 and vanishes outside the ball of radius ϵ about the point 0.

We have seen before thzt holomorphy of a function on a convex set implies the function has a primitive. It is striking to see how the Cauchy integral formula can be used to give the opposite conclusion: If a function is the derivative of a holomorphic function, then it is itself holomorphic.

Corollary 3 *Let Ω be an open subset of $\mathbf{R}^2$. The derivative of a holomorphic function $f : \Omega \to \mathbf{R}^n$ is again holomorphic.*

Observe that the hypotheses of the corollary do not even require explicitly the derivative of f to be continuous, even less totally differentiable.

Proof: We need to show f' is holomorphic at any given point z_0 in Ω. Pick a small r so that the disc of radius r about z_0 is contained in Ω. Let γ be the positively oriented curve along the circle of radius r about z_0. The function

$$g(z_1) = \int_\gamma \frac{f(z)}{z - z_1}\,dz$$

is complex differentiable in the inetrior of the disc with derivative

$$g'(z_1) = \int_\gamma \frac{f(z)}{(z - z_1)^2}\,dz$$

This we can see by taking the limit of the difference quotients and observing that the limit commutes with the integration. Namely,

$$\frac{1}{z_2 - z_1}\left(\frac{1}{z - z_1} - \frac{1}{z - z_2}\right) = \frac{-1}{(z_1 - z)^2} + \frac{z_2 - z_1}{(z_1 - z)^2(z_2 - z)}$$

converges uniformly on the circle to

$$\frac{1}{(z_1 - z)^2}$$

as z_2 tends to z_1, and hence

$$\frac{f(z)}{z_2 - z_1}\left(\frac{1}{z - z_1} - \frac{1}{z - z_2}\right)$$

converges uniformly on the circle to

$$\frac{f(z)}{(z_1 - z)^2}$$

because f is continuos and therefore bounded on the circle.

By a similar argument we see that the function g has a second derivative

$$g''(z) = \int_\gamma \frac{2f(z)}{(z - z_1)^3}\, dz$$

Can apply the same reasoning to the function

$$h(z_1) = \int_\gamma \frac{1}{z - z_1}\, dz$$

which is therefore also twice complex differentiable. Since $h(z_0) \neq 0$ has we have seen before,

$$f(z_1) = \frac{g(z_1)}{h(z_1)}$$

is defined near z_0 and twice complex differentiable there. This is what we had to prove. □

By induction we see actually that if f is holomorphic it is infinitely often complex differentiable.

Combining with the second miracle gives the following converse of the lemma of Goursat:

Lemma 7 (Morera) *If Ω is an open domain convex domain $f : \Omega \to \mathbf{R}^2$ satisfies*

$$\int_\gamma f(z)\, dz = 0$$

for all triangular curves γ such that the interior of the triangle is contained in Ω, then f is holomorphic in Ω.

Proof: Since holomorphy is a local property, we may assume that Ω is a small ball and thus convex. We have $f = F'$ for some holomorphic F on Ω, Hence f is holomorphic by the previous corollary. □

We will turn to power series in one complex variable. The following application of the Lemma of Morera will play a role:

If Ω is an open subset of $\mathbf{R}^2$, then we say that a sequence of functions $f_n : \Omega \to \mathbf{R}^2$ converges to a function $f : \Omega \to \mathbf{R}^2$ uniformly on compact sets, if for every compact subset $K \subset \Omega$ we have

$$\lim_{n\to\infty} \|f_n - f\|_{K,\infty} = 0$$

where

$$\|g\|_{K,\infty} = \sup_{z\in K} |g(z)|$$

Lemma 8 *Let $f_n : \Omega \to \mathbf{R}^2$ be a sequence of holomorphic functions which converge uniformly on compact sets to a function $f : \Omega \to \mathbf{R}^2$. Then f is holomorphic.*

Proof: Let γ be any triangular curve such that the triangle spanned by this curve is contained in Ω. Since the image of γ is compact, f_n converges uniformly to f on the image of γ, f is continuous there, and

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz = 0$$

Since γ was arbitrary, the lemma of Morera implies that f is holomoprhic. □

6 Power series

Observe that polynomials in z are holomorphic functions on all of $\mathbf{R}^2$.

Given a sequence $(a_k)_{k=0,1,\ldots}$ and a point z_0 we can form a sequence of holomorphic functions $f_n : \mathbf{R}^2 \to \mathbf{R}^2$

$$f_n(z) = \sum_{k=0}^{n} a_n (z - z_0)^n \tag{22}$$

Lemma 9 *Let $(a_k)_{k=0,1,\ldots}$ be a sequence of complex numbers and let R be a positive number such that the sequence*

$$|a_k| R^k$$

is bounded. Let $z_0 \in \mathbf{R}^2$ and

$$\Omega = \{z : |z - z_0| < R\}$$

Then the sequence of functions f_n as in (22), restricted to the ball Ω, converges uniformly on compact sets. Moreover, f is holomorphic on Ω.

Proof: Let K be a compact set in Ω. Then there is a $r < R$ such that $z \in K$ implies $|z - z_0| < r$.

We show that f_n is a Cauchy sequence with respect to the norm $\|.\|_{K,\infty}$. But we have for $n < m$

$$\|f_n - f_m\|_{K,\infty} = \| \sum_{k=n+1}^{m} a_k (z - z_0)^k \|_{K,\infty}$$

$$\leq \sum_{k=n+1}^{m} |a_k| r^k \leq C \sum_{k=n+1}^{\infty} \left(\frac{r}{R}\right)^k$$

$$\leq C \left(\frac{r}{R}\right)^n \frac{1}{1 - \frac{r}{R}}$$

The right hand side tends to 0 as n tends ot ∞. Thus f_n is a Cauchy sequence. Therefore f_n converges uniformly on K to a function f. Since in particular a single point is a compact set, f_n converges pointwise everywhere in Ω. Let f be the limit of f_n on Ω.

By the previous corollary to the lemma of Morera f is holomorphic.

□

Given a sequence (a_k), the supremum of all nonnegative numbers R such that $|a_k| R^k$ is bounded is called the radius of convergence of the sequence f_k. It might be 0, a positive real number, or ∞. If R is the radius of convergence and

$$\Omega = \{z : |z - z_0| < R\}$$

then f_n converges uniformly on compact sets in Ω, because each compact set is contained in an open disc about z_0 whose radius r makes $|a_k| r^k$ bounded. (Observe we cannot simply apply the lemma to the radius of convergence R, because $|a_k| R^k$ may not be bounded.

Inside the radius of convergence we write

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

for the limit f and call this the power series associated to the sequence of coefficients $(a_k)_k$.

We have the following uniqueness result:

Lemma 10 *Let z_0 be a complex number and let $z_1, z_2, \ldots$ a sequence of complex numbers in the punctured disc $B_r(z_0) \setminus \{z_0\}$ which converge to z_0 as $n \to \infty$. If*

$$\sum_{k=1}^{\infty} a_k (z - z_0)^k$$

$$\sum_{k=1}^{\infty} b_k (z - z_0)^k$$

are two power series which converge in $B_r(z_0)$ *and satisfy*

$$\sum_{k=1}^{\infty} a_k(z_n - z_0)^k = \sum_{k=1}^{\infty} b_k(z_n - z_0)^k$$

for $n = 1, 2, \ldots$, *then* $a_k = b_k$ *for all* k.

Proof: By taking differences we may assume that $b_k = 0$ for all k. Assume to get a contradiction that not all a_k are zero. Let K be the smallest index for which $a_K \neq 0$. Consider the series

$$\sum_{k=0}^{\infty} a_{k+K}(z - z_0)^k \quad . \tag{23}$$

it has a radius of convergence which is greater or equal the radius of convergence of the series associated to a_k because if

$$|a_k|R^k$$

is bounded then

$$|a_{k+K}|R^k = (R^{-K})|a_{k+K}|R^{k+K}|$$

is bounded.

Thus the power series (23) converges in $B_r(z_0)$.

We have for each $n = 1, 2, \ldots$

$$(z_n - z_0)^K \sum_{k=0}^{\infty} a_{k+K}(z_n - z_0)^k = \sum_{k=0}^{\infty} a_{k+K}(z_n - z_0)^{k+K} = 0$$

Since $z_n \neq z_0$ we obtain

$$\sum_{k=0}^{\infty} a_{k+K}(z_n - z_0)^k = 0$$

Since this power series is continuous at $z = z_0$, it vanishes at z_0. This is a contradiction to $a_{k+K} \neq 0$. $\square$

We have the following existence theorem for power series expansion

Lemma 11 *Assume* $\Omega \subset \mathbf{R}^2$ *is open and* $z_0 \in \Omega$. *Let* D *be the open disc of radius* r *about* z_0 *and assume that the closure of* D *is contained in* Ω. *Then there is a power series about the point* z_0 *with radius of convergence at least* r *which coincides with* f *on* D.

Proof: By a translation we may assume without loss of generality that $z_0 = 0$. Let γ be the curve $[0, 2\pi] \to \Omega$

$$\gamma(t) = (r\cos t, r\sin t)$$

Consider the integral

$$\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Observe that for $z \in D$ and $\zeta \in \partial D$ (boundary of D) we have

$$\frac{1}{\zeta - z} = \frac{1}{\zeta(1 - z/\zeta)}$$

$$= \frac{1}{\zeta} \sum_{k=0}^{\infty} \left(\frac{z}{\zeta}\right)^k = \frac{1}{\zeta} \sum_{k=0}^{\infty} \left(\frac{z}{r^2}\right)^k \overline{\zeta}^k$$

The power series with coefficients $a_k = \left(\frac{z}{r^2}\right)^k$ has radius of convergence $\frac{r^2}{|z|}$ which ius strictly larger than r. Hence this power series converges uniformly on the disc $|\overline{\zeta}| = r$ and we have

$$\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_\gamma \frac{1}{\zeta} \sum_{k=0}^{\infty} \left(\frac{z}{r^2}\right)^k \overline{\zeta}^k \, d\zeta$$

$$= \sum_{k=0}^{\infty} \int_\gamma \frac{1}{\zeta} \left(\frac{z}{r^2}\right)^k \overline{\zeta}^k \, d\zeta = \sum_{k=0}^{\infty} z^k \int_\gamma f(\zeta)\zeta^{-k-1}\, d\zeta$$

This is a power series whose radius of convergence is greater of equal to r, because

$$r^k \left| \int_\gamma f(\zeta)\zeta^{-k-1}\, d\zeta \right| \le r^k \int_0^{2\pi} |f(\gamma(t))| r^{-k-1} | |\gamma'(t)|\, dt$$

$$= \int_0^{2\pi} |f(\gamma(t))|\, dt$$

is bounded.

We apply this reasoning for the function $f = 1$. Let $k > 0$ and observe

$$I_k = \int_\gamma \zeta^{-k-1}\, d\zeta = 0$$

because on $\mathbf{R}^2 \setminus \{0\}$ the function

$$\zeta^{-k-1}$$

has the primitive $-\frac{1}{k}\zeta^{-k}$. Thus all terms of the power series expansion except for the term $k = 0$ vanish. The term $k = 0$ has been calculated before, and we obtain for $z \in D$

$$\int_\gamma \frac{1}{\zeta - z}\, d\zeta = 2\pi i$$

By Cauchy's integral formula we obtain for $z \in D$

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

The previous arguments give a convergent powers series for $f(z)$ in D with coefficients

$$\frac{1}{2\pi i} \int_\gamma f(\zeta)\zeta^{-k-1}\, d\zeta$$

□

We have the following alternative description of of the power series:

Lemma 12 *If $f : \Omega \to \mathbf{R}^2$ is holomorphic at $z_0 \in \Omega$, then the n-th coefficient of the powers series expansion of f about a point z_0 is given by*

$$\frac{f^{(n)}(z_0)}{n!}$$

Proof: Without loss of generality let $z_0 = 0$. We observe that we can termwise differentiate power series. Namely, let

$$\sum_{k=0}^{\infty} a_k z^k, \sum_{k=0}^{\infty} b_k z^k$$

be the unique power series expansions of f and f', both convergent in a small disc D about 0, then for z in D we have

$$\sum_{k=1}^{\infty} a_k z^k = \int_\gamma \sum_{k=0}^{\infty} b_k \zeta^k \, d\zeta = \sum_{k=0}^{\infty} \int_\gamma b_k \zeta^k \, d\zeta = \sum_{k=0}^{\infty} \frac{b_k}{k+1} z^{k+1} =$$

where γ is a straight line form 0 to z. Here we have used that the power series of f' converges unifomly on a sufficiently small neighborhood of z_0. Thus $a_n = nb_{n-1}$, which proves that the series of f' is obtained by term-wise differentiation of the series for f. By using induction and evaluating the power series for $f^{(n)}$ at 0 we obtain the lemma.
□

Thus the unique power series of a holomorphic function at a given point is given by its Taylor series:

$$f(z - z_0) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \quad .$$

Recall that an open set is called connected, if it is not the disjoint union of two nonempty open sets. Existence and uniqueness of power series gives the following uniqueness result for holomorphic functions:

Lemma 13 *Let Ω be a connected open set in $\mathbf{R}^2$. If the power series of two holomorphic functions $f_1 : f_2 : \Omega \to \mathbf{R}^2$ coincide in a point $z_0 \in \Omega$ then $f_1 = f_2$*

By taking the differenece of f_1 and f_2 it suffices to prove that if all coefficients of the Taylor series of a holomorphic function $f : \Omega \to \mathbf{R}^2$ are zero at some point $z \in \Omega$, then f is constant 0 on Ω. This is not an obvious statement, because we are not guaranteed that f coincides on all of Ω with its Taylor series at a given point.

Let Ω' be the set of points $z \in \Omega$ such that all coefficients of the Taylor series of f about z vanish. Then Ω' is an open set, because if $z \in \Omega'$, then the Taylor series of f converges to f on a small neighborhood of z. Therefore f vanishes in a small open neighborhood of z, and this neighborhood is a subset of Ω'.

Now define Ω_k to be the set of all $z \in \Omega$ such that $f^{(k)}(z) \neq 0$. Since $f^{(k)}$ is continuosu, Ω_k is open.

However, $Omega \setminus \Omega'$ is the union of the sets Ω_k, and therfore also open. By connectedness Ω' is empty or equal to Ω. This proves the lemma. □

By the uniqueness result of power series, we may replace the condition of identical; Taylor series at a point by the condition that the two functions coincide on a set of points which has an accumulation point in Ω. Namely, then the Taylor series at this point coincide.

We thus have a strengthening of the third complex miracle:

Complex Miracle 3 *If the domain of a holomorphic function is connected, then the function is determined by the behaviour near an arbitrary point in the domain.*

The third complex miracle says that the behavior of a holomorphic function near a point determines the behavior far away from the point and vice vers.

We shall discuss a striking application of this principle.

Recall that the third miracle materializes in two formulas for the k-th coefficient of the power series expansion of a holomorphic function f about a point z_0, namely

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_\gamma f(\zeta)\zeta^{-k-1}\,d\zeta$$

where γ is a curve describing a circle of radius R about the point z_0.

One can use this identity to obtain size estimates for the derivatives of f:

$$\frac{|f^{(n)}(z_0)|}{n!} \leq \frac{1}{2\pi}R^{-k} \sup_{z\in\partial B_r(z_0)} |f(\zeta)| \tag{24}$$

Most notably we have the following

Theorem 2 *An entire funtion of polynomial growth is a polynomial*

Here a complex valued function is called entire if it is defined and holomorphic on all of $\mathbf{R}^2$, and it is called of polynomial growth if there exist a contant C and a number n such that

$$|f(z)| \leq C(1+|z|)^n$$

Proof: If fg is of polynomial growth with parameters C and n, then the above inequality gives for every radius R:

$$|f^{(n+1)}(z_0)| \leq \frac{n!}{2\pi}R^{-n-1}(C(1+R)^n)$$

The right hand side tends to 0 as R tends to infinity. Hence $f^{(n+1)}$ is constant 0. Hence all higher than $n+1$- st derivatives of f are zero, so its Taylor series about 0 is a polynomial. By uniqueness the entire function f coincides with this polynomial. □

The special case $n = 0$ in the proof of this lemma gives the Theorem of Liouville:

Theorem 3 *Bounded entire functions are constant*

A striking consequence of this theorem is the fundamental theorem of algebra:

Theorem 4 *If*

$$f(z) = \sum_{k=0}^{n} a_k z^k$$

is a non-constant complex polynomial, then there is a $z_0 \in \mathbf{R}^2$ *with* $f(z_0) = 0$.

Proof: By deleting unnecessary terms if necessary, we may assume that $a_n \neq 0$. Also, $n > 0$ because the polynomial was constant otherwise.

We assume to get a contradiction that the polynomial f has no zero. Then

$$\frac{1}{f(z)}$$

is an entire function, because of the chain rule and the fact that the reciprocal $z \to \frac{1}{z}$ is complex differentiable away from the origin.

Let R be equal to

$$\frac{1}{|a_n|}\left(1 + \sum_{k=0,\ldots,n} |a_k|\right)$$

which is a positive number. Then we have for $|z| \geq \max(1, R)$

$$|f(z)| \geq |a_n||z|^n - \sum_{k=0}^{n-1} |a_k||z|^k$$

$$\geq |a_n||z|^n - \sum_{k=0}^{n-1} |a_k||z|^{n-1} \geq R^{n-1}\left(|a_n|R - \sum_{k=0}^{n-1} |a_k|\right) = R^{n-1}$$

On the compact set $\{z : |z| \leq R\}$ the continuous function

$$\left|f(z)^{-1}\right|$$

has a maximum M. Thus we have for all $z \in \mathbf{R}^2$

$$\left|f(z)^{-1}\right| \leq \max(M, R^{n-1})$$

By Liouville's theorem, f^{-1} is constant. This contradicts the hypothesis that f is not constant. □

As a corollary, we observe

Corollary 4 *If* $n \geq 1$ *and*

$$f(z) = \sum_{k=0}^{n} a_k z^k$$

is a polynomial with $a_n \neq 0$, *then there are complex numbers* $z_1, \ldots, z_n$ *such that*

$$f(z) = a_n \prod_{k=1}^{n} (z - z_k)$$

Proof: We prove this statement by induction on n. For $n = 1$ it is obvious because

$$a_0 + a_1 z = a_1 \left(z - \left(-\frac{a_0}{a_1} \right) \right)$$

Now let $n > 1$ and

$$f(z) = \sum_{k=0}^{n} a_k z^k$$

be such a polynomial. Let z_1 be a root of the polynomial which we know to exist by the fundamental theorem of algebra. The Taylor series about the point z_1 is a polynomial of degree n, because the n-th derivative of f at z_1 is nonzero whereas the higher derivatives are 0. Hence the Taylor series is of the form

$$f(z) = \sum_{k=0}^{n} b_k (z - z_1)^k$$

and we observe $a_n = b_n$. Since $f(z_1) = 0$ we observe $b_0 = 0$. hence

$$f(z) = (z - z_1) \sum_{k=0}^{n-1} b_{k+1} (z - z_1)^k$$

Applying the induction hypothesis to the polynomial

$$\sum_{k=0}^{n-1} b_{k+1} (z - z_1)^k$$

proves the lemma. □

We obtain the following corollary for real polynomials:

Corollary 5 *Every polynomial of order $n \geq 1$ with real coefficients can be written as a product of polynomials with real coefficients, each of which having order 0, 1 or 2.*

Proof: We prove this statement again by induction on n. The case $n = 1$ is trivial. let $n > 0$. If $f(z) = 0$, we are immediately done by applying induction hypothesis on

$$\sum_{k=0}^{n-1} a_{k+1} z^k$$

If $f(z) \neq 0$, we split the polynomial as

$$a_n \prod_{k=1}^{n} (z - z_k)$$

If z_1 is real, then

$$a_n \prod_{k=2}^{n} (z - z_k)$$

is a real polynomial (we can calculate its derivatives at 0 to be real numbers) and we can use induction hypothesis on it. If z_1 is not real, then $\overline{z_1}$ is also a root of the polynomial and we may assume that $z_2 = \overline{z_1}$. Then

$$(z - z_1)(z - z_2) = z^2 - (z_1 + \overline{z_1})z + (z_1\overline{z_1})$$

is a real polynomial and either

$$f(z) = \alpha_2(z - z_1)(z - z_2)$$

or we can use the induction hypothesis on the real polynomial

$$a_n \prod_{k=3}^{n} (z - z_k)$$

□

We can now prove an assertion made earlier that **C** is the only field which is a finite dimensional vector space over **R**.

Theorem 5 *Let $n \geq 2$ and assume we have a product*

$$\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$$

which makes $\mathbf{R}^n$ a field such that

$$(x, 0, \ldots, 0)(x_1, x_2, \ldots, x_n) = (xx_1, xx_2, \ldots, xx_n) \tag{25}$$

*Then $n = 2$ and there is a bijective field homomorphism from **C** to this field.*

Remark: In the setting of real vector spaces it is natural to assume that **R** can be imbedded linearly into the field ($x \in \mathbf{R}$ goes to x times the unit of the field), and then the hypothesis (25) is satisfied in general by an appropriate change of coordiates.

Proof:

Let S be the set of all $(x, 0, \ldots, 0)$ with $x \in \mathbf{R}$. Pick $y \in \mathbf{R}^n \setminus S$. With respect to the given product consider the elements

$$y^0, y^1, y^2, \ldots, y^n$$

Since these are $n+1$ elements in $\mathbf{R}^n$, there are real coefficients $\alpha_0, \alpha_1, \ldots, \alpha_n$, not all of them 0, such that

$$\sum_{k=0}^{n} \alpha_k y^k = 0$$

By (25) this can be read as a linera combination in the vector space $\mathbf{R}^n$ or as a polynomial identity in the field $\mathbf{R}^n$.

By the corollary to the fundamental theorem of algebra we can split this polynomial as a product of polynomials with real coefficients of order 0, 1, and 2. Since in a field a product vanishes if and only if one of the factors vanishes, we observe that

x is the root of a real polynomial of degree at most 2. In fact the degree is exactly 2 because x is not in S.

Let $b_0 + b_1 y + b_2 y^2$ be this real polynomial. We may assume $b_2 = 1$. Let z be a complex root of this polynomial.

Then z is not real. We claim that the injective linear map σ from $\mathbf{R}^2$ to $\mathbf{R}^n$ mapping 1 to $(1, 0, \ldots, 0)$ and z to x is a homomorphism of fields.

Since we have linearity by construction, it suffices to prove

$$\sigma(z_1 z_2) = \sigma(z_1)\sigma(z_2)$$

for all complex numbers z_1 and z_2. By the distributive law and linearity it suffices to prove this for z_1 and z_2 being 1 or z. If either one of them is equal to 1, the identity follows trivially, so it suffices to prove

$$\sigma(z^2) = x^2$$

Hovever, we have

$$\sigma(z^2) = \sigma(-b_0 - b_1 z) = -b_0(1, 0, \ldots, 0) - b_1 x = x^2$$

This proves that σ is a field homomorphism.

Now we claim that σ is surjective. Namely, any y in $\mathbf{R}^n \setminus S$ is the root of a real quadratic polynomial, which we may assume to have leading coefficient 1. We can write this polynomial in $\mathbf{R}^n$ as

$$y^2 - \sigma(z_1 + z_2)y + \sigma(z_1 z_2) = (y - \sigma(z_1))(y - \sigma(z_2))$$

where z_1 and z_2 are the complex roots of the polynomial. Thus the root y is either $\sigma(z_1)$ or $\sigma(z_2)$, which proves the claim. □

7 The exponential function

The complex numbers $\mathbf{C}$ form a group under addition. The set $\mathbf{C} \setminus \{0\}$ forms a group under multiplication. Thus it is natural to study group homomorphisms

$$f : \mathbf{C} \to \mathbf{C} \setminus \{0\} \tag{26}$$

of these groups. Such homomorphisms f satisfy

$$f(z_1 + z_2) = f(z_1) f(z_2) \tag{27}$$

for all $z_1, z_2 \in \mathbf{C}$. We shall look for entire functions f satisfying (26) and (27).

Let f be such an entire function, then setting $z_1 = z_2 = 0$ we obtain

$$f(0) = f(0)^2$$

which, since $f(0) \neq 0$ by (26) implies

$$f(0) = 1$$

Moreover, differentiation of (27) with respect to z_1 gives

$$f'(z_1 + z_2) = f'(z_1)f(z_2)$$

inserting $z_1 = 0$ and setting $c = f'(0)$ we obtain

$$f'(z) = cf(z)$$

for all $z \in \mathbf{C}$.

Thus an entire homomorphism f as above satisfies an initial value problem

$$f(0) = 1, \quad f'(z) = cf(z) \tag{28}$$

We shall first consider the case $c = 1$. Suppose f is a solution of this initial value problem. Since f is holomorphic, it is given by its Taylor series in a small neighborhood of 0 (Recall $0! = 1$ and $k!$ is the product of the numbers $1, \ldots, k$ for $k > 0$)

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$$

Here we have used that by induction all derivatives of f are equal to f. Thus we have seen that the Taylor series about 0 is determined by the initial value problem, and the uniqueness result (third miracle) shows that the initial value problem has at most one solution in any connected open set containing the origin.

The radius of convergence of the above series is infinite, because for every integer N and every n we have

$$\frac{N^n}{n!} \leq N^N N!$$

which is easily seen both for $n \leq N$ and $n > N$. Thus this power series defines an entire function f. By construction the derivative of f is equal to itself, hence we have proved existence of an entire solution to the initial value problem (28) with $c = 1$.

We define

$$\exp(z) := \sum_{k=0}^{\infty} \frac{1}{k!} z^k$$

For general values of the parameter c, we see by similar arguments that $f(z) = \exp(cz)$ is the unique solution to the initial value problem (28).

Thus $\exp(cz)$ are the only candidates for entire homomorphisms. We prove that they are indeed entire homomorphisms.

We consider the case $c = 1$. First observe that exp has no zero. Namely, assume to get a contradiction that $\exp(z_0) = 0$. Then all derivatives of exp at z_0 vanish, and the Taylor series of exp about z_0 is constant equal to 0. By uniquness (third miracle) this implies exp is constant 0 everywhere, which is a contradiction to $\exp(0) = 1$. Thus exp has no zeroes.

If z' is a complex number, we observe that the function

$$g(z') = \frac{\exp(z + z')}{\exp(z')}$$

(here we use the fact $\exp(z') \neq 0$, which has just been established), satisfies the above initial value problem with parameter 1. By uniqueness

$$\exp(z') = g(z') = \frac{\exp(z + z')}{\exp(z')}$$

Hence exp is a homomorphism.

For the case of general c we conclude immediately from the case $c = 1$ that

$$\exp(cz) \neq 0, \quad \exp(c(z_1 + z_2)) = \exp(cz_1)\exp(cz_2)$$

Hence $f(z) = \exp(cz)$ is a homomorphism for all c. Observe that for $c = 0$ we obtain the trivial homomorphism $f = 1$.

Thus we have established the following:

Lemma 14 *The entire homomorphisms*

$$f : \mathbf{C} \to \mathbf{C} \setminus \{0\}$$

from the additive to the multiplicative group of $\mathbf{C}$ *are given by the functions*

$$f(z) = \exp(cz)$$

For different c *these functions are different.*

Proof: It only remains to observe that for different c the functions $\exp(cz)$ are different. This however follows immediately from the fact that the derivative of $\exp(cz)$ at 0 is c. □

Observe that if we drop the requirement of being entire, we have more homomorphisms from the additive to the multiplicative group. namely, if c and d are complex numbers, then

(29) $$f(a + ib) = \exp(ca)\exp(db)$$

is easily seen to be a homomorphism, which is entire only if $d = ic$. It can be seen that (29) are the only continuous homomorphisms from the additive to the multiplicative group of $\mathbf{C}$.

We shall discuss the values of the function exp. Since all coefficients of the Taylor series of exp are real, we have

$$\exp(\overline{z}) = \sum_{k=0}^{\infty} a_k \overline{z}^k$$

$$= \sum_{k=0}^{\infty} \overline{a_k z^k} = \overline{\sum_{k=0}^{\infty} a_k z^k}$$
$$= \overline{\exp(z)}$$

for all z.

In particular f takes real values on the real axis. For $t \geq 0$, we see immediately

$$\exp(t) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \geq 1 + t$$

In particular exp is positive for $t \geq 0$ and tends to ∞ as $t \to \infty$. All derivatives of exp are equal to exp, hence they are positive for $t \geq 0$ and in particular exp is monotone increasing and convex for $t \geq 0$.

Using $\exp(-t)\exp(t) = 1$ we see that $\exp(t)$ is positive for all real t and tends to 0 as $t \to -\infty$. By the same argument as before exp is monotone and convex for all real t. Also, we observe that

$$\exp(t) \geq 1 + t$$

holds for all $t \in \mathbf{R}$ because exp is convex and tangent to $1 + t$ at $t = 0$.

We shall now consider exp for purely imaginary arguments, i.e. $\exp(it)$ for real t.

We have

$$|\exp(it)|^2 = \exp(it)\overline{\exp(it)} = \exp(it)\exp(-it) = \exp(0) = 1$$

Thus $\exp(it)$ is a complex number on the unit circle.

For $t > 0$ Consider the curve

$$\gamma : [0, t] \to \mathbf{R}^2, \gamma(s) = \exp(is)$$

At each point $\exp(is)$ its tangent vector is $i\exp(is)$, which is a vector tangent to the circle at $\exp(is)$ and points counterclockwise. Thus the curve runs monotonously counterclockwise along the circle. The length of the curve is

$$\int_0^t |\gamma'(s)|\, ds = \int_0^t 1\, ds = t$$

A similar discussion holds for negative t, the curve now running clockwise.

Thus the point $\exp(it)$ has angle t from the real axis, which together with $|\exp(it)| = 1$ gives a complete description of the values of exp at purely imaginary arguments.

In particular

$$\exp it = 1$$

if and only if t is an integer multiple of 2π. Given that, we observe that exp is periodic with period $2\pi i$:

$$\exp(z + 2\pi i) = \exp(z)\exp(2\pi i) = \exp(z)$$

For general complex arguments, we observe

$$\exp(a + ib) = \exp(a)\exp(ib)$$

thus $\exp(ab)$ has distance $\exp(a)$ from the origin and angle b to the real axis.

8 More on the exponential function, logarithm

We will write

$$e^z$$

instead of $\exp(z)$. Namely by the property

$$\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$$

for integer arguments z, the value $\exp(z)$ coincides with the z-th power of the number

$$e = \exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!}$$

Moreover, for a rational $\frac{p}{q}$, the value $\exp(\frac{p}{q})$ coincides with the porsitive q-th root of $\exp(p)$. By contnuity, for real values of z, $\exp(z)$ coincides with the standard definition of z-th porwer of e.

We can understand the exponential map by factoring it as follwos

$$\mathbf{R} \times \mathbf{R} \to \mathbf{R} \times (\mathbf{R}/2\pi\mathbf{Z}) \to \mathbf{R}^2 \setminus \{0\} \tag{30}$$

Here the first map maps the plane to the infinite cylinder $\mathbf{R} \times (\mathbf{R}/2\pi\mathbf{Z})$ by mapping (x, y) to $(x, [y])$ where $[y]$ denotes the set (equivalence class) of all y' in $\mathbf{R}$ which differ from y by an integer multiple of 2π. The second map maps

$$(x, [y]) \to e^x e^{iy}$$

It is well defined, because e^{iy} is independent of the representative $y + 2\pi n$ of the equivalence class $[y]$. Moreover, this second map is a bijection. Namely, assume

$$e^x e^{iy} = e^{x'} e^{iy'}$$

Taking absolute values we observe

$$e^{x-x'} = 0$$

and in turn

$$e^{i(y-y')} = 0$$

Since e^z is strong monotone for real values of z, we conclude $x - x' = 0$. By replacing y by a different representative of the class $[y]$ we may assume $0 \leq y - y' < 2\pi$. By our geometric desription of $e^{i(y-y')}$ as having angle $y - y'$ from the real axis, we conclude $y - y' = 0$.

In fact, the map exp is what is called a universal covering map for the space $\mathbf{R}^2 \setminus \{0\}$. In the setting of holomorphic functions this property can be phrased as follows

Lemma 15 *Let Ω be an open convex subset of $\mathbf{R}^2$ Let $f : \Omega \to \mathbf{R}^2 \setminus \{0\}$ be a holomorphic map. Then there is a holomorphic map $g : \Omega \to \mathbf{R}^2$ such that*

$$f(z) = e^{g(z)}$$

for all $z \in \Omega$.

Proof: Since f has no zero, we can define a holomorphic function

$$\frac{f'}{f}$$

on Ω. By the second miracle, this function has a holomorphic primitive g on Ω:

$$g' = \frac{f'}{f}$$

(We use that Ω is convex.). By adding a constant to g we may assume

$$f(0) = e^{g(0)}$$

Now consider the function

$$h(z) = \frac{1}{f(z)} e^{g(z)}$$

on Ω. It's derivative is equal to

$$h'(z) = -\frac{f'(z)}{f(z)^2} e^{g(z)} + \frac{g(z)}{f(z)} e^{g(z)} = 0$$

Thus h is constant. Since $h(0) = 1$ we have that h is constant equal to 1. This proves that g has the required properties.□

Moreover, the exponential function is essentially unique with this property, namely we have

Lemma 16 *Let $h : \mathbf{R}^2 \to \mathbf{R}^2 \setminus \{0\}$ be a holomorphic map with the following property: For every holomorphic map $f : \mathbf{R}^2 \to \mathbf{R}^2 \setminus \{0\}$ there exists a holomorphic map $g : \mathbf{R}^2 \to \mathbf{R}^2$ such that*

$$f(z) = h(g(z))$$

for all $z \in \mathbf{R}^2$. Then there are complex numbers $a \neq 0$ and b such that

$$h(z) = e^{az+b}$$

Proof: homework exercise.

We conclude our discussion of the exponential function by some remarks on the concept of exponential function in the setting of Lie groups.

For the purpose of this discussion a Lie group is a closed subgroup of the group $Gl(n, \mathbf{R})$ of invertible real $n \times n$ matrices, the group structure given by matrix multiplication. E.g., the multiplicative group of $\mathbf{R}^2 \setminus \{0\}$ can be regarded as the group of invertible 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Another important example ofcourse is the group $Gl(n, \mathbf{R})$ itself.

Such a closed subgroup G of $Gl(n, \mathbf{R})$ can be regarded as a subset of $\mathbf{R}^{n^2}$. As such we can study differentiable curves

$$\gamma : \mathbf{R} \to G$$

The tangent space T at the unit element I of G (the identity matrix) is given by the set of all vectors

$$\left.\frac{d\gamma}{dt}\right|_{t=0}$$

where γ is any curve with $\gamma(0) = I$. It can be seen that this tangent space is a real vector space, e.g., in the case of the particular group above this tangent space has dimension 2.

Moreover, it can be seen, that for each tangent vector t there is a unique curve $\gamma_t : \mathbf{R} \to G$ which has tangent vector t at I and is a group homormoprhism:

$$\gamma_t(r + r') = \gamma_t(r)\gamma_t(r')$$

Here the product on the right hand side is the product of the Lie group, which is given by matrix multiplication. Such a curve is called a one parameter group. It can be found as follows:

$$\gamma_t(r) = \lim_{n \to \infty} \left(I + r\frac{t}{n} \right)^n$$

(Here the tangent vector t, which is an element of $\mathbf{R}^{n^2}$, is understood to be an $n \times n$ matrix.) The map $t \to \gamma_t(1)$ is called the exponential map of the Lie group. It specializes to the exponential map of complex numbers in the situation that the Lie group is the multiplicative group of the complex numbers.

For the complex exponential map observe

Lemma 17 *The sequence*

$$f_n(z) = \left(1 + \frac{z}{n}\right)^n$$

of holomorphic functions converges uniformly on compact sets to the functions e^z.

Proof: homework exercise

9 Logarithms

We try to find an inverse to the exponential function.

1) Right inverse: we try to find a holomorphic function f so that

$$\exp \circ f(z) = z$$

whenever z is in the domain of f. We shall assume the domain Ω of f is open (as usual) and connected. The domain of f cannot contain 0, because exp never takes the value 0.

We observe that if f is such a right inverse, then $f + 2\pi i$ is another such inverse, so f can be determined at most up to multiples of $2\pi i$.

Taking derivatives of the above identity yields

$$f'(z)e^{f(z)} = f'(z)z = 1$$

Since 0 is not in the domain Ω, we obtain

$$f'(z) = \frac{1}{z}$$

Vice versa, if f satisfies $f'(z) = 1/z$, then the deriuvative of the function

$$\frac{1}{z}e^{f(z)}$$

is easily seen to be zero. Hence the function itself is constant, and by adding an appropriate constant to f we can assume this constant is equal to 1. Thus f is a right inverse of exp.

2) Left inverse: We try to find a holomorphic function f so that

$$f \circ \exp(z) = z$$

whenever $z \in \Omega'$ for some suitable open Ω'. Thus the domain Ω of f should contain the values $\exp(z)$ with $z \in \Omega'$. Since $\exp(z) = \exp(z + 2\pi i)$ this identity cannot be simultaneously satisfied for z and z' if $z - z'$ is a multiple of $2\pi i$. Hence we have to assume Ω' contains only one representative of each class $z + 2\pi i\mathbf{Z}$. We may assume 0 is not in the domain Ω' of f because exp never vanishes. Moreover, we shall assume Ω to be open and connected (by restricting Ω' if necessary).

Differentiating the above identity gives

$$\exp(z)f'(\exp(z)) = 1$$

Hence

$$f'(z) = \frac{1}{z}$$

whenever z is of the form $\exp(z')$ with $z' \in \Omega$. It is easy to see that this set of z is large enough so that by the third miracle we can conclude $f'(z) = 1/z$ in all of Ω.

Vice verse, if $f(z) = 1/z$ in the domain of z, then, where defined, the function

$$f(e^z)$$

has vanishing derivative and therefore is constant. By adding an appropriate constant to f, this constant is 0.

The above considerations show that we have to look for primitives of $\frac{1}{z}$.

The maximal domain on which we could hope for such a primitive is $\mathbf{R}^2 \setminus \{0\}$. However, such a primitive cannot exist, because if it did we would conclude

$$\int_\gamma \frac{1}{z} \, dz = 0$$

for all closed curves in the domain. But we have seen that this integral is $2\pi i$ if γ describes a counter-clock-wise circles of radius r about the origin.

However, by the second complex miracle we can find a primitive of $1/z$ on every convex domain Ω which does not contain 0. In fact, the proof we did gives a primitive on the slit plane

$$\mathbf{R}^2 \setminus \{rz : r \geq 0\}$$

for any $z \neq 0$. Such a primitive f with appropriate choice of a constant, so that

$$z = \exp(f(z))$$

wherever defined is called a (branch) of the logarithm.

For further discussion of this topic see homework assignments.

10 The Riemann sphere

Taking quotients of functions becomes problematic whenever the function in the denominator has the zeros.

However, holomorphic functions have only very sparse zero sets. We will see that as a consequence one can make sense of quotients of holomorphic functions without restriction:

Complex Miracle 4 *Quotients of holomorphic functions make sense and can be interpreted again as functions.*

Let Ω be an open connected subset of $\mathbf{R}^2$. Let $H(\Omega)$ denote the set of all holomorphic functions $f : \Omega \to \mathbf{R}^2$.

Then $H(\Omega)$ is a complex vector space, and has a product (pointwise product of functions) which makes it a commutative algebra. Namely, the product satisfies associative, distributive, commutive laws and has a unit element, which is the function that is constant equal to 1.

The space $H(\Omega)$ fails to be a field, because all functions in $H(\Omega)$ which have a zero do not have an inverse in $H(\Omega)$.

However, we have

Lemma 18 *The algebra $H(\Omega)$ is an integral domain, i.e., if $f, g \in H(\Omega)$ and $fg = 0$ then $f = 0$ or $g = 0$.*

Proof: Assume $fg = 0$. Pick $z_0 \in \Omega$ and a sequence $z_n \in \Omega$ which converges to z_0 and satisfies $z_n \neq z_0$ for all n. Then for each n we have $f(z_n) = 0$ or $g(z_n) = 0$. By symmetry we may assume there are infinitely many n such that $f(z_n) = 0$. Then f vanishes on a set with accumulation point z_0, so by the thrid miracle f is the zero function. □

The algebra $H(\Omega)$ being an integral domain, we can define the quotient field of $H(\Omega)$.

Namely, consider the set of all pairs

$$(f, g)$$

with $f, g \in H(\Omega)$ and $g \not\equiv 0$, and define such a pair (f_1, g_1) to be equivalent to the pair (f_2, g_2), if $f_1 g_2 = f_2 g_1$. We write $(f_1, g_1) \sim (f_2, g_2)$ if (f_1, g_1) is equivalent to (f_2, g_2).

This notion of equivalence gives an equivalence relation in the mathematical sense. Namely, it satisfies reflexivity ($(f, g) \sim (f, g)$), symmetry ($(f_1, g_1) \sim (f_2, g_2)$ implies $(f_2, g_2) \sim (f_1, g_1)$, and transitivity ($(f_1, g_1) \sim (f_2, g_2)$ and $(f_1, g_1) \sim (f_3, g_3)$ imply that $(f_2, g_2) \sim (f_3, g_3)$). Reflexivity and symmetry are easy to see. To see transitivity we have to use that $H(\Omega)$ is an integral domain. Namely, $f_1 g_2 = f_2 g_1$ and $f_1 g_3 = f_3 g_1$ imply

$$g_1 f_3 g_2 = f_1 g_3 g_2 = f_1 g_2 g_3 = g_1 f_2 g_3$$

Hence

$$g_1(f_3 g_2 - f_2 g_3)$$

Since $g_1 \not\equiv 0$ and $H(\Omega)$ is an integral domain we conclude

$$f_3 g_2 = f_2 g_3$$

which had to be shown.

Now the quotient field of $H(\Omega)$ is the set of all equivalence classes of pairs $(f, g) \in H(\Omega \times (H(\Omega) \setminus \{0\})$. We shall write

$$\frac{f}{g}$$

for such an equivalence class. Sums and products of such classes are defined by

$$\frac{f_1}{g_1} + \frac{f_2}{g_2} = \frac{f_1 g_2 + f_2 g_1}{g_1 g_2}, \quad \frac{f_1}{g_1} \frac{f_2}{g_2} = \frac{f_1 f_2}{g_1 g_2}$$

Ofcourse, one has to check that these definitions do not depend on the special choise of the pair (f, g) in the class $\frac{f}{g}$.

This procedure is analoguous to passing to the quotient field $\mathbf{Q}$ from the integral domain $\mathbf{Z}$. It explains on an abstract level that one is able to make sense of quotients of holomorphic functions.

But an even stronger statement is true: One can make sense of the quotient of two elements in $H(\Omega)$ as a function on Ω. At points where the denominator does not vanish, one can just take the quotient of the values of the function, but if the denominator vanishes one has to be more careful. Indeed, one has to extend the range of possible values of functions to include a point called infinity.

Consider the space $\mathbf{R}^3$ and the unit sphere S therein, i.e., the set of all points (x, y, t) with $x^2 + y^2 + t^2 = 1$.

There is a map from $\mathbf{R}^2$ to S defined by

$$\mu : (x, y) \to \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right)$$

One easily checks that μ indeed maps into S. Moreover, $\mu(x, y)$ lies on the line from the north pole $(0, 0, 1)$ to the point $(x, y, 0)$, and μ is geometrically determined by this requirement.

One also checks that the image of μ is the sphere without the north pole $(0, 0, 1)$, and that μ has an inverse map mapping $S \setminus \{(0, 0, 1)\}$ to the plane.

As a subset of $\mathbf{R}^3$ the sphere S inherits a metric: the distance of two points in the sphere is the distance of the two points as elements in $\mathbf{R}^3$. This makes the sphere a topological space. As a bounded closed subset of $\mathbf{R}^3$ the sphere S is compact. The map μ and its inverse on $S \setminus \{(0, 0, 1)\}$ are continuous.

The one-point-compactification of a topological space A is a topological space $A \cup \{\infty\}$, where ∞ is a symbol for a point not in A, and the open sets of $A \cup \{\infty\}$ are given by the open sets U of A and the sets $U \cup \{\infty\}$ where U is the complement of a compact set in A. If we extend the map μ to a map

$$\mu^* \mathbf{R}^2 \cup \{\infty\} \to S$$

by setting $\mu^*(\infty) = (0, 0, 1)$, then μ^* becomes a homeomorphism of the one point compactification $\mathbf{R}^2 \cup \{\infty\}$ to S, i.e., a continuous bijection with a continuous inverse.

The map $\mu = (\mu_1, \mu_2, \mu_3)$ is conformal in the following sense: Its Jacobian

$$J = \begin{pmatrix} \frac{\mu_1}{x} & \frac{\mu_2}{x} & \frac{\mu_3}{x} \\ \frac{\mu_1}{y} & \frac{\mu_2}{y} & \frac{\mu_3}{y} \end{pmatrix}$$

$$= \frac{2}{(1 + x^2 + y^2)^2} \begin{pmatrix} 1 + y^2 - x^2 & -2xy & 2x \\ -2xy & 1 - y^2 + x^2 & 2y \end{pmatrix}$$

Satisfies

$$JJ^T = \frac{4}{(1 + x^2 + y^2)^2} I$$

and the determinant of the matrix

$$\begin{pmatrix} \frac{\mu_1}{x} & \frac{\mu_2}{x} & \frac{\mu_3}{x} \\ \frac{\mu_1}{y} & \frac{\mu_2}{y} & \frac{\mu_3}{y} \\ -\mu_1 & -\mu_2 & -\mu_3 \end{pmatrix} \tag{31}$$

is always a positive real number. The latter condition is an orientation condition, it is the analoguous condition that conformal maps in two dimensions have positive Jacobian determinant. Observe that (32) can be interpreted as the Jacobian of the map

$$(x, y, z) \to e^{-z}\mu(x, y)$$

on the plane $z = 0$.

This condition of conformality means that if

$$\gamma_1 : [-1, 1] \to \mathbf{R}^2$$

$$\gamma_2 : [-1, 1] \to \mathbf{R}^2$$

are two curves in $\mathbf{R}^2$ with $z_0 = \gamma_1(0) = \gamma_2(0)$, then the angle between the tangent vectors ${\gamma_1}'(0)$ and ${\gamma_2}'(0)$ at z_0 (viewed from above the plane) is the same as the angle between the tangent vectors $(\mu \circ \gamma_1)'(0)$ and $(\mu \circ \gamma_2)'(0)$ at $\mu(z_0)$ (viewed from the inside of the sphere). In particular, μ preserves orthogonality of tangent vectors.

11 Meromorphic functions

If $\Omega \subset \mathbf{R}^2$ is an open set, we define a map $\nu : \Omega \to S$,

$$(x, y) \to (\nu_1(x, y), \nu_2(x, y), \nu_3(x, y))$$

to be conformal, if it is differentiable, its Jacobian J_ν satisfies $J_\nu(z)J_\nu^T(x, y) = c(x, y)I$ for some $c(x, y) \neq 0$ and the determinant of the matrix

$$\begin{pmatrix} \frac{\nu_1}{x} & \frac{\nu_2}{x} & \frac{\nu_3}{x} \\ \frac{\nu_1}{y} & \frac{\nu_2}{y} & \frac{\nu_3}{y} \\ -\nu_1 & -\nu_2 & -\nu_3 \end{pmatrix} \tag{32}$$

is non-negative. (A remark on our terminology: Most text books define conformal so that the Jacobian is required to be non-degenerate. We have chosen to include the case of singular Jacobian in the definition of conformality.) Observe that in particular conformal maps int S are continuous.

Observe that if $f : \Omega \to \mathbf{R}^2$ is holomorphic, then $\mu \circ f$ is conformal. Vice versa, if $f : \Omega \to \mathbf{R}^2$ is a map and $\mu \circ f$ is conformal, then f is holomorphic.

A function $f : \Omega \to \mathbf{C} \cup \{\infty\}$ is called meromorphic, if $\mu \circ f$ is conformal. In particular, if f is meromorphic and $f(z_0) \neq \infty$ for some point $z_0 \in \Omega$, then f is holomorphic when restricted to a suitable small neighborhood of z_0.

We observe that for a complex number $z \neq 0$, $\mu(z)$ and $\mu(z^{-1})$ differ by a rotation by π about the x- axis. Namely

$$\mu(z^{-1}) = \mu\left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right)$$

$$= \left(\frac{2x}{1+x^2+y^2}, -\frac{2y}{1+x^2+y^2}, -\frac{x^2+y^2-1}{x^2+y^2+1} \right)$$

$$= \mu(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Thus it makes sense to define

$$\frac{1}{0} = \infty$$

$$\frac{1}{\infty} = 0$$

With this definition, we have a characterization of meromorphic maps which does not refer to the Riemann sphere:

Lemma 19 *Let $f : \Omega \to \mathbf{C} \cup \{\infty\}$ be a map. Assume that f is holomorphic at all $z \in \Omega$ for which $f(z) \neq \infty$ and that f^{-1} is holomorphic at all $z \in \Omega$ for which $f(z) \neq 0$. Then f is meromorphic.*

Vice versa, if f is meromorphic, then it is holomorphic at all z for which $f(z) \neq \infty$, and f^{-1} is holomorphic for all z for which $f(z) \neq 0$.

Proof: This is immediate from what has been said before, observing that if ν is a conformal map, then the composition of ν with a rotation of the sphere is a conformal map. □

Our goal is to identify quotients of holomorphic maps as meromorphic functions. For this it is convenient to have the notion order of a zero of a holomorphic function.

Lemma 20 *Given a holomorphic function $f : \Omega \to \mathbf{R}^2$ which is not constant to zero and $z_0 \in \Omega$, there is a unique integer $n \geq 0$ and a unique holomorphic function $g : \Omega \to \mathbf{R}^2$ with $g(z_0) \neq 0$ such that*

$$f(z) = (z - z_0)^n g(z)$$

To see existence, consider the coefficients a_k of the power series expansion of f about z_0 and let n be the smallest integer so that $a_n \neq 0$. Then

$$f(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k$$

$$= (z - z_0)^n \sum_{k=0}^{\infty} a_{k+n} (z - z_0)^k$$

and the power series

$$\sum_{k=0}^{\infty} a_{k+n} (z - z_0)^k \tag{33}$$

in the last expression is easily seen to have the same radius of convergence as the power series of f. Thus this powers series defines a holomorphic function in a neighborhood of z_0, which coincides with

$$g(z) := \frac{f(z)}{(z - z_0)^n} \tag{34}$$

on the intersection of $\Omega \setminus \{z_0\}$ with the domain of convergence (33). Defining g by (34) on $\Omega \setminus z_0$ and $g(z_0) = a_n$ proves the existence statement.

The above calculation relating the power series expansion of g and f holds whenever

$$f(z) = (z - z_0)^n g(z)$$

hence uniqueness of the power series expansion shows uniqueness of n and g. □

The number n in this lemma is called the order of the the holomorphic function f at z_0.

We now can identify the quotient of holomorphic maps as meromorphic functions. Observe that given two holomorphic maps $f, g : \Omega \to \mathbf{R}^2$, we can define their quotient pointwise at all points which are not a zero of g.

Lemma 21 *The quotient of two holomorphic maps $f, g : \Omega \to \mathbf{R}^2$ with g not constant 0 can be uniquely extended to a meromorphic function on Ω.*

Proof: Observe that the zero set of g consists of isolated points.

Since being meromorphic is a local property, it suffices to consider a single zero z_0 of g and prove there is a unique element in $\mathbf{R}^2 \cup \{\infty\}$ which we can assign to z_0 to make the extended map $\frac{f}{g}$ meromorphic in z_0. Uniqueness is not a question, because meromorphic maps are continuous and $\mathbf{R}^2 \cup \{\infty\}$ is Hausdorff.

Given z_0, let Ω' be a neighborhood of z_0 such that f and g do not vanish on $\Omega' \setminus \{z_0\}$. Here we use again that the zeros of f and g have no accumulation points. Write

$$f(z) = (z - z_0)^n \tilde{f}(z)$$

$$g(z) = (z - z_0)^m \tilde{g}(z)$$

where $\tilde{f}$ and $\tilde{g}$ are holomorphic on Ω' and have no zeros in Ω'.

If $n \geq m$ we define h on Ω' by

$$h(z) = (z - z_0)^{n-m} \tilde{f}(z) \tilde{g}^{-1}(z)$$

This is a holomorphic extension of $\frac{f}{g}$ to z_0.

If $n < m$ we define h on Ω' by

$$h(z) = (z - z_0)^{m-n} \tilde{f}^{-1}(z) \tilde{g}(z)$$

Then h^{-1} is a meromorphic extension of $\frac{f}{g}$ in Ω'. This proves the lemma.

□

12 Meromorphic functions as quotients of holomorphic functions

We have seen that a function $\Omega \to \mathbf{R}^2 \cup \{\infty\}$ is meromorphic if and only if f is holomorphic at all points $z \in \Omega$ with $f(z) \neq \infty$ and $1/f$ is holomorphic at all points $z \in \Omega$ with $f(z) \neq 0$. A point z_0 is called a pole of f if $f(z_0) = \infty$.

Given two holomorphic function $f, g : \Omega \to \mathbf{R}^2$, both not constant 0, we can write unambiguously

$$f(z) = (z - z_0)^n \tilde{f}(z)$$
$$g(z) = (z - z_0)^m \tilde{g}(z)$$

where $\tilde{f}$ and $\tilde{g}$ are holomorphic and do not vanish at z_0. Then we have seen that we can extend f/g meromorphically in z_0 by setting $f/g(z_0)$ equal to 0, $\tilde{f}(z_0)/\tilde{g}(z_0)$ or ∞ if $n > m$, $n = m$, or $n < m$ respectively.

Thus quotients of holomorphic functions can be extended meromorphically, and this extension is unique.

We shall now study te converse: are meromorphic fucntions always quotients of holomorphic maps? We will prove the answer to be affirmative if the domain is $\mathbf{R}^2$.

First of all, locally this statement is always true. Namely, locally a meromorphic function can be written as f or $1/(1/f)$, where f or $1/f$ is holomorphic. Thus the question is of global nature.

Observe that we can extend the notion of order of a function to meromorphic functions. We define the order of a meromorphic function f at z_0 to be n if f is holomorphic and has order n, and we define the order to be $-n$ if $1/f$ is holomorphic and has order n. It is easy to see that if f has order n and g has order m at z_0, then fg has order $n + m$ at z_0.

Given a meromorphic function f, our goal is to find a holomorphic function which at each pole of f of order $-n$ has a zero of order at least n. Then $h := fg$ will be holomorphic everywhere, and $f = h/g$.

Let Ω be open and connected and $f : \Omega \to \mathbf{R}^2 \cup \{\infty\}$ meromorphic. We observe that, as for holomorphic functions, no point $z_0 \in \Omega$ is acculmulation point of zeroes of f, unless f is constant 0.

Namely, consider the set of accumulation points of zeroes of f inm Ω. It is open, because near an accumulation point of zeros f is holomorphic and therfore constant 0. Moreover, it is closed by basically definiution. By connectedness, this set is either Ω or empty.

By taking reciprocals, we see that no point of Ω is accumulation point of poles.

We would like to find nonzero holomorphic functions, say on $\mathbf{R}^2$, which have prescribed zero sets. If there are finitely many zeros, a polynomial will do. If there are infinitely many zeros, the infinite product

$$\prod_{k=1}^{\infty} (1 - \frac{z}{z_k})$$

may not converge. The idea of Weierstrass is to replace the factors in this product by factors which are close to 1 near the origin. For a positive integer consider

$$E_p(z) = (1-z)\exp\left(z + \frac{z^2}{2} + \ldots + \frac{z^p}{p}\right)$$

The argument of the exponential function is the beginning of the Taylor series of $-\log(1-z)$. Thus one expects this function to be close to 1 where this approximation of $-\log(1-z)$ is good, which is for $|z|<1$. Precisely we have

$$E_p'(z) = -z^p \exp\left(z + \frac{z^2}{2} + \ldots + \frac{z^p}{p}\right)$$

Thus

$$\frac{1-E_p(z)}{z^{p+1}}$$

is holomorphic and - as can easily be seen - has a Taylor series whose coefficients are all positive numbers. Thus, for $|z|<1$, we have

$$\left|\frac{1-E_p(z)}{z^{p+1}}\right| \leq 1 - E_p(1) = 1$$

We need some abstract lemmas on infinite products:

Lemma 22 *Let $c_1, \ldots, c_n$ be a complex numbers with*

$$\sum_{k=1}^{\infty} |c_k| \leq C$$

Then

$$\left|\left[\prod_{k=1}^{n}(1+c_k)\right] - 1\right| \leq Ce^C$$

Proof:

$$\left|\left[\prod_{k=1}^{n}(1+c_i)\right] - 1\right| = \left|\int_0^1 \frac{d}{dt}\prod_{k=1}^{n}(1+tc_i)\,dt\right|$$

$$\leq \int_0^1 \left(\sum_{k=1}^{n}|c_k|\right)\prod_{k=1}^{n}(1+|c_k|)\,dt \leq Ce^C$$

□

Lemma 23 *Let Ω be an open subset of $\mathbf{R}^2$ and let $c_n : \Omega \to \mathbf{R}^n$ be a sequence of holomorphic functions such that*

$$\sum_{k=1}^{\infty} \|c_k\|_{\infty,K}$$

is finite for every compact set $K \subset \Omega$ Then

$$f_n(z) = \prod_{k=1}^{n} (1 + c_k(z))$$

converges uniformly on compact sets to a holomorphic function $f : \Omega \to \mathbf{R}^n$ which vanishes at a point $z \in \Omega$ if and only if one of the f_n vanishes at z.

Fix a compact set K. By splitting off finitely many factors, we may assume that on K

$$\sum_{k=1}^{\infty} \|c_k(z)\|_{\infty,K} \leq 1/4$$

and prove that the series of partial products converges uniformly to a nowhere vanishing function.

However, we have that f_n is Cauchy w.r.t. $\|.\|_{\infty,K}$. Namely, for $n > m$ we have

$$\left\| \prod_{k=1}^{n} (1 + c_k) - \prod_{k=1}^{m} (1 + c_k) \right\|_{\infty,K}$$

$$\leq \left\| \prod_{k=1}^{m} (1 + c_k) \right\|_{\infty,K} \left\| \left[\prod_{k=m+1}^{n} (1 + c_k(z)) \right] - 1 \right\|_{\infty,K}$$

$$\leq \left(\frac{e^{1/4}}{4} + 1 \right) e^{1/4} \left[\sum_{k=m+1}^{n} \|c_k\|_{\infty,K} \right]$$

This clearly proves f_n is Cauchy. Moreover, we have

$$\left\| \left[\prod_{k=1}^{n} (1 + c_k) \right] - 1 \right\|_{\infty,K} \leq \frac{e^{1/4}}{4} < 1$$

Which proves that the limit f of f_n does not vanish in K. □

Now we apply these abstract lemmata to products of functions $E_p(z/z_n)$:

Lemma 24 *Let Ω be $\mathbf{R}^2$. Let z_n be a sequence in Ω with possible repetition but no accumulation point. Then there is a holomorphic function on Ω which has a zero at every z_n but nowhere else (the multiplicity of vanishing is the same as the mutliplicity of z_n in the sequence)*

Proof: We may assume the sequence $|z_n|$ is non-decreasing. Also, observe that since z_n has no accumulation point, we have for each z a n_0 such that $|z_n| > |z|$ for $n > n_0$.

Define a sequence of integers p_n such that

$$\sum_{k=1}^{\infty} \left(\frac{|z|}{|z_n|} \right)^{p_n}$$

is finite for all $z \in \mathbf{R}^n$. For example if we pick p_n so that $|z_n|^{p_n} > n!$ whenever $|z_n| > 1$ we will have such a sequence.

Then consider the infinite product

$$\prod_{k=1}^{\infty} E_{p_n}(\frac{z}{z_n})$$

Let K be a closed disk of radius r about the origin. Consider n_0 such that $|z_n| > r$ for $n > n_0$. Then the product of the first m factors with $m > n_0$ has the wanted zeros with correct orders inside the disc. Thus it suffices to prove that there is m such that the infinite product

$$\prod_{k=m}^{\infty} E_{p_n}(\frac{z}{z_n})$$

exists and does bot vanish in K. By the previous lemmata it suffices to prove

$$\sum_{k=m}^{\infty} \|E_{p_n}(\frac{z}{z_n}) - 1\|_{\infty,K} < \frac{1}{2}$$

However, the left hand side is bounded by

$$\sum_{k=m}^{\infty} \left(\frac{r}{|z_n|}\right)^{p_n}$$

which can be made small by sufficiently large choice of m. □

The following corollary is the desired conclusion

Lemma 25 *Let Ω be $\mathbf{R}^2$ of the unit disc about 0. If $f : \Omega \to \mathbf{R}^2 \backslash \{0\}$ is meromorphic, then there exist two holomorphic functions g, h on Ω such that $f = h/g$*

Proof: Let z_n be the sequence of poles of f, each pole appearing $-n$ times if n is the order of f at the pole. Construct a nonzero function g as in the previous lemma that has zeros at the points z_n. Then $h = fg$ is holomorphic. □

The above proof can be easily adapted to show that every meromorphic function on Ω is a quotient of holomrophic functions if ω is a disc, or, with a bit more effort, any connected open subset of $\mathbf{R}^2$.

Not only it makes sense to consider conformal mappings into the Riemann sphere, but also conformal mappings from the Riemann sphere. We shall imemdiately pass to a description not involving the Riemann sphere: as we did for meromorphic functions.

Definition 3 *If Ω is an open subset of $\mathbf{R}^2 \cup \{\infty\}$, then a map*

$$f : \Omega \to \mathbf{R}^2$$

is called holomorphic if f is holomorphic at all points $z \neq \infty$ and, if $\infty \in \Omega$, then the function

$$z \to f\left(\frac{1}{z}\right)$$

defined in a neighborhood of 0*, is holomorphic at* 0*. Likewise one defines a map*

$$f : \omega \to \mathbf{R}^2 \cup \{\infty\}$$

to be meromorphic if f is meromorphic at all points $z \neq \infty$ and, if $\infty \in \Omega$, then the function

$$z \to f\left(\frac{1}{z}\right) \quad ,$$

defined in a neighborhood of 0*, is meromorphic at* 0*.*

There are no holomorphic functions on the Riemann sphere other than the constant functions. Namely, the Riemann sphere is compact, so the image of such a holomorphic map was compact, hence bounded. Thus the restriction of this map to $\mathbf{R}^2$ is constant by the theorem of Liouville. Thus f is constant on the Riemann sphere.

Polynomial functions

$$\sum_{k=0}^{n} a_k z^k$$

are meromorphic functions from the Riemann sphere to the Riemann shere. Namely, they are holomorphic in $\mathbf{R}^2$ and

$$z \to \sum_{k=0}^{n} a_k z^{-k}$$

is clearly meromorphic at 0.

In particular we see that for the Riemann sphere, not all meromorphic functions are quotients of holomorphic functions.

Rational functions are quotients of polynomial functions. As such they are clearly meromorphic functions on the Riemann sphere. We shall see that they are the only meromorphic functions on the Riemann sphere. Let f be a meromorphic function on the Riemann sphere. Since there are no accumulation points of poles of f and the Riemann sphere is compact, there are only finitely many poles. Thus there is a polynomial g such that fg has at most a pole at ∞. Then $(fg)^{-1}$ has no zeros other than possibly at infty, and it has again finitely many poles. We can choose another polynomial h so that $h(fg)^{-1}$ has no zeros and no poles other than at ∞. Then either $h(fg)^{-1}$ is holomorphic or $h^{-1}(fg)$ is holomorphic. In either case $h(fg)^{-1}$ is constant, which proves that f is a rational function.

13 Residue calculus, Casorati Weierstrass

The following result is a nice criterion to decide whether a function, holomorphic in a punctured neighborhood $\Omega \setminus \{z_0\}$ of z_0 can be extended meromorphically to z_0.

Lemma 26 (Casorati-Weierstrass) *Let Ω be open and $z_0 \in \Omega$. A holomorphic function $f : \Omega \setminus \{z_0\}$ can be extended to Ω meromorphically if and only if there is a neighborhood $\Omega' \subset \Omega$ of z_0 such that the image of $\Omega' \setminus \{z_0\}$ is not dense in $\mathbf{R}^2$.*

Proof: If f can be extended meromorphically, let $f(z_0)$ be the value of the extended function. Pick an open neighborhood U of $f(z_0)$ in $\mathbf{R}^2 \cup \{\infty\}$ which is not dense in $\mathbf{R}^2 \cup \{\infty\}$. By continuity of f, there is a neighborhood Ω' of z_0 which is mapped into U. Hence $f(\omega' \setminus \{z_0\})$ is not dense in $\mathbf{R}^2$.

Now assume there is a neighborhood Ω' so that $A = f(\Omega' \setminus \{z_0\})$ is not dense in $\mathbf{R}^2$. We can find r and z_0 soi that the ball of radius r about z_1 is not contained in A. Then

$$g(z) = \frac{1}{f(z) - z_1}$$

is bounded in Ω'. Moreover $g(z)(z - z_0)$ can be continuously extended to z_0 by setting $g(z_0) = 0$. This continuous extension is holomorphic near z_0 by the lemma of Goursat. By taking quotients we conclude that $f(z)$ has a meromorphic extension to z_0. □

If a meromorphic function has a pole at z_0 of order $-n$, then we can write

$$f(z) = (z - z_0)^{-n} \tilde{f}(z)$$

for some meromorphic $\tilde{f}$ which is neither 0 nor ∞ at z_0

Expanding $\tilde{f}$ into a power series near z_0 gives

$$f(z) = (z - z_0)^{-n} \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

$$= \sum_{k=-n}^{\infty} b_k (z - z_0)^k$$

where $b_k = a_{k+n}$ for all $k \geq -n$. The sum

$$\sum_{k=-n}^{-1} b_k (z - z_0)^k$$

is called the principal part of the function f near z_0. The remaining series

$$\sum_{k=0}^{\infty} b_k (z - z_0)^k$$

is easily seen to be a convergent Taylor series near z_0. The *residue* of f at z_0 is defined to be

$$\mathrm{Res}(f, z_0) = b_{-1}$$

If $\gamma : [a, b] \to \mathbf{R}^2$ is a closed curve, then the winding number of γ around a point z_0 not in the image of γ is defined as

$$\mathrm{Ind}(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} \, dz$$

Lemma 27 *Let Ω be a convex domain in $\mathbf{R}^2$ and let $f : \Omega \to \mathbf{R}^n$ be meromorphic. Let $\gamma : [a,b] \to \Omega$ be a piecewise continuously differentiable closed curve. whose image does not contain a pole of γ. Then there are finitely many poles $z_1, \ldots, z_n$ of f in the convex hull of $\gamma([a,b])$ and*

$$\int_\gamma f(z)\, dz = \sum_{k=1}^{n} 2\pi i \mathrm{Res}(f, z_k) \mathrm{Ind}(\gamma, z_k)$$

Proof: The convex hull K of $\gamma([a,b])$ is closed and bounded and therefore compact. It is a subset of Ω because Ω is convex. Moreover, there is an open neighborhood Ω' of K such that all poles of f in Ω' are inside K.

By restricting f we may assume $\Omega = \Omega'$. Let f_k be the principal part of f at z_k. Then

$$f - \sum_{k=1}^{n} f_k$$

is holomorphic at all z_k, hence it is holomorphic in Ω. Thus

$$\int_\gamma f(z)\, dz = \sum_{k=1}^{n} \int_\gamma f_k(z)\, dz$$

Since $(z - z_k)^{-n}$ for $n > 2$ has a primitive in $\mathbf{R}^2 \setminus \{z_0\}$ we have

$$\sum_{k=1}^{n} \int_\gamma f_k(z)\, dz = \sum_{k=1}^{n} \mathrm{Res}(f, z_k) \int_\gamma \frac{1}{z - z_k}\, dz$$

This proves the lemma.

Remark: While we have so far focussed on using the Cauchy integral formula to evaluate functions at a point by calculating the Cauchy integral, we point remark that in the form of the residue calculus the Cauchy integral formula is frequently used to explicitly calculate integrals.

An easy example is given by

$$\lim_{N \to \infty} \int_{-N}^{N} \frac{1}{1+x^2}\, dx$$

The integral can be read as evaluating the complex line integral

$$\int_\gamma \frac{1}{1+z^2}\, dz$$

along the curve

$$\gamma : [-N, N] \to \mathbf{R}^2, t \to (t, 0)$$

By adding a curve

$$\tilde{\gamma} : [0, \pi] \to \mathbf{R}^2, t \to (N\cos(t), N\sin(t))$$

the composed curve becomes closed.

On the enclosed semicircle the integrand

$$f(z) = \frac{1}{1+z^2} = \frac{1}{z-i}\,\frac{1}{z+i}$$

has a pole only at $z = i$. Since the pole is simple, the residue is given by the continuous extension of $f(z)(z-i)$ at i which is $\frac{1}{2i}$. The winding number of the curve $\gamma \cup \tilde{\gamma}$ about i is 1. Hence

Hence

$$\int_\gamma \frac{1}{1+z^2}\,dz + \int_\gamma \frac{1}{1+z^2}\,dz = 2\pi i \frac{1}{2i} = \pi$$

The second integral on the left hand side can be estimated (for $N > 1$) by

$$\left| \int_0^\pi f(\gamma(t))\gamma'(t)\,dt \right| \leq \int_0^\pi \frac{1}{N^2-1} N\,dt \leq \frac{1}{N-1}$$

This tends to 0 as N tends to ∞. Hence

$$\lim_{N\to\infty} \int_{-N}^N \frac{1}{1+x^2}\,dx = 2\pi i \frac{1}{2i} = \pi$$

While in this case the primitive of the integrand can be identified as arcus tangent, so we could evaluate this integral by the fundamental theorem of calculus, often the residue calculus is the only way to evaluate an integral. On a metamathematical level this residue calculus explains why so many explicite values for integrals involve the number π.

Counting zeros and poles

Complex Miracle 5 *One can count zeros of holomorphic functions.*

Let $f : \Omega \to S$ be a meromorphic function. Recall that near a point z_0 we can write

$$f(z) = (z-z_0)^n \tilde{f}(z)$$

where $\tilde{f}$ is meromorphic in Ω and

$$\tilde{f}(z_0) \notin \{0, \infty\}$$

The integer n, positive or negative, is unique and called the order of f at z_0. If n is positive, then we say that f has n zeros at z_0 (multiplicity n). From an analytic point of view one has to count zeros with multiplicities, as one can see from the following example: As z_1 tends to z_0, the function

$$(z-z_1)(z-z_0)$$

tends uniformly on compact sets to the function

$$(z - z_0)^2$$

If the number of zeros in some open set of a function is at all to depend continuously on the function f with respect to uniform convergence on compact sets, then the zero of the second function needs to count as 2 zeros.

Now consider the meromorphic function

$$g(z) := \frac{f'(z)}{f(z)} = \frac{n(z - z_0)^{n-1} f(z) + (z - z_0)^{n-1} \tilde{f}'(z)}{(z - z_0)^n \tilde{f}(z)}$$

$$= \frac{n}{z - z_0} + \frac{\tilde{f}'(z)}{\tilde{f}(z)}$$

Since $\tilde{f}$ does not vanish at z_0, g is meromorphic at z_0 with residue

$$\mathrm{Res}(g, z_0) = n$$

In particular, g is holomoprhic at z_0 if and only if f does has no zero and no pole at z_0.

We can obtain the number n by the integral

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz$$

whereγ is an appropriate clsoed curve with winding number 1 around z_0.

Another way of motivating this calculation is to view f'/f formally as the derivative of $\log f$. ("log-derivative"). We say formally because we do not want to bother with discussing the branch and domain of the logarithm.

Since the logarithm of a product is the sum of the logarithms we expect, and in fact can easily prove by algebra, that

$$\frac{(\prod_{k=1}^n f_i)'}{\prod_{k=1}^n f_k} = \sum_{k=1}^n \left(\frac{f_k{}'}{f_k} \right)'$$

The above formula for $g(z)$ is an immediate corollary of this general formula.

Let $\mathrm{Ord}(f, z_0)$ denote the order of a function f at z_0.

Then we have

Lemma 28 *Let $\Omega \subset \mathbf{R}^2$ be open and convex. Let $f : \Omega \to S$ be a meromorphic function in Ω with finitely many poles and zeros. Let $\gamma : [a, b] \to \Omega$ be a piecewise continuously differentiable closed curve such that f has no poles or zeros in the image of γ. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz = \sum_{z_0} \mathrm{Ord}(f, z_0) \mathrm{Ind}(\gamma, z_0)$$

where the sum goes over all $z_0 \in \Omega$ at which f has a pole or a zero.

As we have discussed for the general residue calculus, the assumption of having finitely many poles and zeros is not a restriction of the general case.

Proof: This is an immediate corollary of the Residue calculus □

The integral

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$$

has a nice geometric interpretation. We can write its as

$$\frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\,dt$$

$$=\frac{1}{2\pi i}\int_a^b \frac{(f\circ\gamma)'(t))}{(f\circ\gamma)(t)}\,dt$$

$$=\frac{1}{2\pi i}\int_{f\circ\gamma}\frac{1}{z}\,dz$$

Thus it is the winding number of the curve $f\circ\gamma$ about the point 0.

Theorem 6 (Rouche) *Let Ω be open and convex in $\mathbf{R}^2$ and let D be a disc in Ω such that the closure of D is contained in Ω.*

Let f and g be two holomorphic functions in Ω such that

$$|f(z)-g(z)|<|f(z)|$$

for all z in the boundary of D. Then f and g have the same number of zeros - counted with multiplicity - in D.

Proof: Let $\gamma=(r\cos t-z_0, r\sin t-z_0)$ be the circular curve describing the boundary of D. For simplicity and without loss of generality we assume $z_0=0$. Observe that the hypothesis of the theorem implies that f and g do not vanish on the boundary of D.

We have to show

$$\int_\gamma \frac{f'(z)}{f(z}\,dz=\int_\gamma \frac{g'(z)}{g(z}\,dz$$

However, we have

$$\int_\gamma \frac{g'(z)}{g(z}-\frac{f'(z)}{f(z}\,dz$$

$$=\int_\gamma \frac{(g/f)'(z)}{(g/f)(z)}\,dz \tag{35}$$

Since

$$\left|\frac{g(z)}{f(z)}(z)-1\right|\le\frac{|g(z)-f(z)|}{|f(z)|}<1$$

on the image of γ, we can use a branch of the logarithm defined in an open disc of radius 1 about 1 to obtain a primitive $\log(f/g)$ of the integrand of (37). Hence (37) vanishes by the fundamental theorem of calculus. □

Geometrically we can interpret Rouche's theorem as follows: Let $f \circ \gamma$ be the path of a person. The person has a dog on a leash the dog walking the curve $g \circ \gamma$. The hypothesis of the theorem says that the leash of possibly variable length is always too short for the dog to reach the origin. Hence the dog walks as often around the origin as the person.

As a further application of lemma 28 we deduce the open mapping and inverse function theorems for holomorphic functions.

A function $f : \Omega \to \mathbf{R}^2$ is called a local homeomorphism at a point $z_0 \in \omega$ if there is a small neighborhood $\Omega' \subset \Omega$ of z_0 such that the restriction $f_{\Omega'}$ of f to Ω' is injective and continuous and the inverse of $f_{\Omega'}$, define don the image of $f_{\Omega'}$ is also continuous.

Theorem 7 *Let $\Omega \subset \mathbf{R}^2$ be open and connected and let $f : \Omega \to \mathbf{R}^2$ be a non-constant holomorphic function. Then $f(\omega)$ is open.*

The function f is a local homeomorphism at a point $z_0 \in \Omega$ if and only if $f'(z_0) \neq 0$

Remark: since the statements of this theorem are of local nature, they also hold with the appropriate interpretation for meromorphic maps on open sets of the Riemann sphere.

Proof:

To prove $f(\Omega)$ is open let $z_0 \in \Omega$. We have to show $f(\Omega)$ contains an open set about z_0. By subtracting $f(z_0)$ from f we may assume $f(z_0) = 0$. Since z_0 is not an accumulation point of zeros, there is a small punctured disc $D \setminus \{z_0\}$ about z_0 on wich f does not vanish. Let γ be a circular curve about z_0 inside D. The image of $f \circ \gamma$ is compact, so its complement is open. Hence there is an open ball D' about 0 which does not intersect the image of $f \circ \gamma$. Hence the winding number $\mathrm{Ind}(f \circ \gamma, z)$ is the same for all $z \in D'$, and is positive since $\mathrm{Ind}(f \circ \gamma, 0) > 0$. (The winding number $\mathrm{Ind}(f \circ \gamma, z)$ counts how often the value z is taken by the function fg inside the circle γ: this is an easy corollary of theorem 28.) Hence all points of D' are in the image of f. This proves that f is open.

Moreover, if z_0 is a simple zero of f, i.e., the derivative of f at z_0 does not vanish, then $\mathrm{Ind}(f \circ \gamma, 0) = 1 = \mathrm{Ind}(f \circ \gamma, z)$ for all $z \in D'$. By continuity of f we can pick a a smaller disc D'' inside D which is mapped into D' by f. The previous discussion shows that f is injective on D'. Thus it has an inverse. The inverse is continuous because we have seen that f maps open sets to open sets. Thus f is a homeomorphism on D'.

Finally, if $f'(z_0) \neq 0$, then

$$\mathrm{Ind}(f \circ \gamma, z) > 1$$

for all z in a small disc about $f(z_0)$, no matter how small the circular arc γ, hence f cannot be locally injective.

□

The Gamma function

Observe that for $t > 0$ we can define

$$t^z := e^{z \log t}$$

where log is a branch of the logarithm on, say, $H = \{z : \Re(z) > 0\}$ which satisfies $\log(1) = 0$.

We define the Gamma function Γ for $\Re(z) > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

For $\Re(z) > 0$ this integral converges in the improper Riemann sense

$$\lim_{n\to\infty} \int_{1/n}^n t^{z-1} e^{-t}\, dt$$

and indeed this convergence is uniform on compact sets in z (exercise). For example we have

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$$

The Gamma function is holomorphic in the positive half plane as one can see from the Lemma of Morera: for any closed curve γ in H we have by Fubini

$$\int_\gamma \int_{1/n}^n t^{z-1} e^{-t}\, dt\, dz$$

$$= \int_{1/n}^n \int_\gamma t^{z-1} e^{-t}\, dz\, dt = 0$$

So Γ is holomorphic as limit uniformly on compact sets of holomorphic functions.

For $\Re(z) > 0$ we observe by partial integration (the boundary terms vanish)

$$z\Gamma(z) = \int_0^\infty z t^{z-1} e^{-t}\, dt = \int_0^\infty t^z e^{-t}\, dt = \Gamma(z+1)$$

Thus

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} \tag{36}$$

Using this functional equation, it follows by induction that

$$\Gamma(n) = (n-1)!$$

for all positive integers n. Thus the Gamma function is an extension of the factorial function defined on the positive integers. It is not the only holomorphic extension, for example we could add

$$\sin(2\pi z) = \frac{1}{2i}\left(e^{2\pi i z} - e^{-2\pi i z}\right)$$

to obtain another extension. However, we will see momentarily that the Gamma function is the most natural extension of the factorial.

Observe that the right hand side of (36) actually makes sense for $\Re(z) > -1$ as a meromorphic function with a simple pole at $z = 0$. Thus the Gamma function has a (unique, as we know) meromophic extension to the half plane $\Re(z) > -1$. Inserting this unique extension on the right hand side of (36) we can extend the Gamma function to $\Re(z) > -2$. This extension has simple poles at $z = 0$ and $z = -1$. By induction Γ has a unique meromorphic extension to the entire complex plane, with poles at $0, -1, -2, \ldots$.

Observe that $\Gamma(z)\Gamma(1-z)$ has simple poles at all integers. Thus it is natural to consider the holomorphic function

$$f(z) := \sin(\pi z)\Gamma(z)\Gamma(1-z)$$

We have the following

Lemma 29 *The function $f(z)$ above is constant and equal to π.*

Proof: We observe that the function f is periodic with period 1:

$$f(z+1) = \sin(\pi z + \pi)\Gamma(z+1)\Gamma(-z)$$

$$f(z+1) = (-\sin(\pi z)\,(z\Gamma(z))\left(\frac{1}{-z}\Gamma(-z+1)\right) = f(z)$$

If log is any branch of the logarithm on any convex subset of $\mathbf{R}^2 \setminus \{0\}$, then consider

$$F(z) = f\left(\frac{\log z}{2\pi i}\right)$$

Since log is determined up to multiples of $2\pi i$, the function F does not depend on the choice of the constant (periodicity of f). Thus, such F defined on two different convex sets coincides on the intersection of these sets. By pasting F together we can define a holomorphic function on all of $\mathbf{R}^2 \setminus \{0\}$.

Our goal is to prove that F is holomorphic at 0 and at ∞. This is done by a simple size estimate for F. We have for $x > 0$:

$$|\Gamma(x+iy)| = \left|\int_0^\infty t^x t^{iy} e^{-t}\,dt\right| \leq \int_0^\infty t^x e^{-t}\,dt = \Gamma(x)$$

We claim that $|\Gamma(x+iy)| \leq |\Gamma(x)|$ also holds for negative x. This follows by induction from

$$|\Gamma(x+iy-1)| = \left|\frac{1}{x+iy-1}\right| |\Gamma(x+iy)|$$

$$\leq \left|\frac{1}{x-1}\right| |\Gamma(x)| = |\Gamma(x+iy)|$$

Therefore

$$|f(x+iy)| \leq |\Gamma(x)\Gamma(1-x)|e^{\pi|y|} \leq Ce^{\pi|y|}$$

Here we have used that $\Gamma(z)\Gamma(1-z)$ as a periodic continuous function on the real line is bounded on the real line. Hence

$$|F(z)| \leq Ce^{\left|\Re\left(\frac{\log z}{2}\right)\right|} \leq \max\left(|z|^{\frac{1}{2}}, |z|^{-\frac{1}{2}}\right)$$

This prove that $F(z)$ is meromorphic at 0 and ∞ (namely, $zF(z)$) is holomorphic at 0 and $F(z)/z$ is holomorphic at ∞ by the general lemma of Goursat.) Moreover, the value of $zF(z)$ at 0 is 0, so $F(z)$ is even holomorphic at 0, and likewise $F(z)$ is holomorphic at ∞. hence F is holomorphic on the entire sphere and therefore is constant.

To obtain the value of this constant observe that the residue of Γ at 0 is equal to $\Gamma(1)$ which is 1. The derivative of $\sin(\pi z)$ at 0 is π. Hence $f(0) = \pi$. □

Corollary 6 *The Gamma function is the only meromorphic function on* $\mathbf{R}^2$ *which satsifies* $\Gamma(1) = 1$, *the functional equation (36), has poles only at the non-positive integers (which then necessarily are simple poles), and can be estimated along each vertical line by the value at the intersection of the vertical line with the real axis:*

$$|\Gamma(x+iy)| \leq C|\Gamma(x)|$$

for all $x, y \in \Re$ *(this is read to be correct if* γ *has a pole at* x*).*

Proof: Observe that in the proof of the previous lemma we have indeed proved

$$\pi = \sin(\pi z)\Gamma_1(z)\Gamma_2(z)$$

for all meromorphic functions Γ_i, $i = 1, 2$ satisfying the properties stated in the corollary. Setting $\Gamma_1 = \Gamma$ and Γ_2 arbitrary we can solve this identity for Γ_2 and conlcude that $\Gamma_2 = \Gamma$. □

Corollary 7 *We have*

$$\pi^{-\frac{1}{2}} 2^{z-1} \Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{z+1}{2}\right) = \Gamma(z)$$

Proof: We need to show the left hand side satisfies the properties of the previous lemma. Clearly it is meromorphic with the right poles. Also it takes its maximal values along vertical lines at the real axis (the function 2^{1-z} has constant modulus along vertical lines). The value at $z = 1$ is 1 because we can solve the identity

$$\Gamma\left(\frac{1}{2}\right) \Gamma\left(1 - \frac{1}{2}\right) \sin\left(\frac{\pi}{2}\right) = \pi$$

for $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. It remains to show the functional equation, which is done as follows

$$\pi^{-\frac{1}{2}} 2^{z+1-1} \Gamma\left(\frac{z+1}{2}\right) \Gamma\left(\frac{(z+1)+1}{2}\right)$$
$$= 2\pi^{-\frac{1}{2}} \frac{2^{z-1}}{\Gamma} \left(\frac{z+1}{2}\right) \frac{z}{2} \Gamma\left(\frac{z}{2}\right)$$
$$= z\pi^{-\frac{1}{2}} 2^{z-1} \Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{z+1}{2}\right)$$

□

The Riemann ζ function

The Riemann ζ function is defined for $\Re(z) > 1$ by

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

Indeed, if $x > 1$ is the real part of z, this sum is convergent by the integral comparison test

$$\sum_{n=1}^{\infty} |n^{-z}| = \sum_{n=1}^{\infty} n^{-x} \leq$$

$$\int_{t=\frac{1}{2}}^{\infty} t^{-x}\, dt = \frac{2^{x-1}}{x-1} \tag{37}$$

This argument also shows easily that the sum defining ζ converges uniformly on compact sets on the half plane $\Re(z) > 1$. Thus ζ is a holomorphic function in that half plane.

Looking at the integral on the left of (37), we observe that while the integral is not convergent for $x < 1$, its value on the right hand side of (37) has a meromorphic extension to the entire plane with a simple pole at $x = 1$. We aim to prove the more subtle fact that ζ has a meromorphic extension to the entire plane too.

First we find a different representation of the zeta function. Some of the justifications below are easier for $\Re(z) > 2$, so we shall for simplicity restrict attention to that domain. We have

$$\Gamma(z)\zeta(z) = \sum_{n=1}^{\infty} n^{-z} \int_0^{\infty} t^z e^{-t}\, \frac{dt}{t} = \sum_{n=1}^{\infty} \int_0^{\infty} t^z e^{-nt}\, \frac{dt}{t}$$

A formal calculation changing the order of summation and integration gives

$$\Gamma(z)\zeta(z) = \int_0^{\infty} \sum_{n=1}^{\infty} t^{z-1} e^{-nt}\, dt = \int_0^{\infty} t^{z-1} \frac{e^{-t}}{1-e^{-t}}\, dt = \int_0^{\infty} \frac{t^{z-1}}{e^t - 1}\, dt$$

To justify changing of the order of integration and summation observe

$$\Gamma(z)\zeta(z) = \lim_{m\to\infty} \sum_{n=1}^{\infty} n^{-z} \int_0^{m} t^{z-1} e^{-t}\, dt$$

inside the limit we can change the order of integration and summation because the partial sums of the integrand converge uniformly, hence

$$\Gamma(z)\zeta(z) = \lim_{m\to\infty} \int_0^{m} \frac{t^{z-1}}{e^t - 1} dt$$

This limit is easily seen to exist, and hence is equal to the improper Riemann integral.

Clearly it is enough to prove that the function $\Gamma(z)\zeta(z)$ has a meromorphic extension. To this end, we shall derive a different expression for $\Gamma(z)\zeta(z)$.

Consider the branch log of the logarithm defined on the slit plane

$$\mathbf{R}^2 \setminus \{it : t \leq 0\}$$

such that $\log(1) = 0$. Then we consider the holomorphic function

$$f(s) = \frac{e^{(z-1)\log s}}{e^{-s} - 1} \tag{38}$$

on this slit plane. For $0 < r < 2\pi$ we consider the curves $\gamma_1 : [r, 2\pi] \to \mathbf{R}^2$, $\gamma_2 : [0, \pi] \to \mathbf{R}^2$ and $\gamma_3 : [r, \infty) \to \mathbf{R}^2$ defined by

$$\gamma_1(t) = r + 2\pi - t$$

$$\gamma_2(t) = r\cos(t) + ir\sin(t)$$

$$\gamma_3(t) = -t$$

These curves can be composed to a curve $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$.

The function f has no poles other than possibly at $s = 2\pi i n$ for n a non-negative integer. Thus the line integral

$$I_+ = \int_\gamma f(s)\, ds \tag{39}$$

does not depend on r for $0 < r < 2\pi$ (old homework exercise).

For $\Re(z) > 2$ the integrand f can be continuously extended to 0 by setting $f(0) = 0$. Thus we have

$$\lim_{r \to 0} \int_{\gamma_2} f(s)\, ds = 0$$

Thus (39) is given by

$$-\int_0^{2\pi} \frac{e^{(z-1)\log t}}{(e^{-t} - 1)}\, dt - \int_0^\infty \frac{e^{(z-1)\pi i} e^{(z-1)\log t}}{(e^t - 1)}\, dt$$

Similarly we can consider the complex conjugate curve $\overline{\gamma}$ and the function f as in (38) with a branch of the logarithm on the slit plane

$$\mathbf{R}^2 \setminus \{it : t \geq 0\}$$

with $\log(1) = 0$. Then the corresponding integral

$$I_- = \int_{\overline{\gamma}} f(s)\, ds$$

is again independent of the choice of $0 < r < 2\pi$ and from $r \to 0$ we obtain

$$I_- = -\int_0^{2\pi} \frac{e^{(z-1)\log t}}{(e^{-t} - 1)}\, dt - \int_0^\infty \frac{e^{-(z-1)\pi i} e^{(z-1)\log t}}{(e^t - 1)}\, dt$$

Subtracting the two integrals we obtain

$$I_+(z) - I_-(z) = \left(-e^{i\pi(z-1)} + e^{-i\pi(z-1)}\right)\Gamma(z)\zeta(z)$$

Thus

$$\Gamma(z)\zeta(z) = \frac{I_-(z) - I_+(z)}{2i\sin(\pi z)}$$

However, the integrals I_+ and I_- are defined for all z in the complex plane, because we have sufficiently rapid decay of the integrands as γ or $\overline{\gamma}$ tend to ∞. Moreover, the integrals are holomorphic in the entire plane, as one can see from the lemma of Morera. Thus $\Gamma(z)\zeta(z)$ can be extended meromorphically to the entire plane.

We now calculate for $\Re(z) < -2$ the integrals I_+ by the residue calculus. Observe that the function f has simple poles at the points $2\pi ik$ where k runs through the positive integers. The residue is

$$\begin{aligned}\mathrm{Res}(f, 2\pi ik) &= -e^{(z-1)\left(\frac{\pi i}{2} + \log 2\pi k\right)} \\ &= ie^{\frac{\pi iz}{2}}(2\pi k)^{z-1}\end{aligned}$$

We now increase the radius r in the definition of the curves γ beyond 2π. Let I_+^n be the integral (39) where the semicircle γ_2 goes around n singularities of f in the strict upper half plane. Then

$$\begin{aligned}I_+^n &= I_+ + \sum_{k=1}^{n} 2\pi i \mathrm{Res}(f, 2\pi ik) \\ &= I_+ - (2\pi)^z e^{\frac{\pi iz}{2}} \sum_{k=1}^{n} k^{z-1}\end{aligned}$$

With the same argument for I_- we obtain

$$I_+ - I_- = I_+^n - I_-^n + (2\pi)^z 2i\sin\left(\frac{\pi z}{2}\right)\sum_{k=1}^{n} k^{z-1}$$

Now we can choose r to be $2\pi i(n + \frac{1}{2})$. Then $|e^{-s} - 1| > \epsilon > 0$ for all s on γ_2. Since $\Re(z) < -2$ we see that

$$\int_{\gamma_2} f(s)\, ds$$

tends to 0 as r tends to ∞. So does

$$\int_{\gamma_3} f(s)\, ds$$

because of the rapid growth of $e^{-s} - 1$. Since the integrals over γ_1 cancel in the differnece $I_+^n - I_-^n$ we obtain

$$I_+ - I_- = (2\pi)^z 2i\sin\left(\frac{\pi z}{2}\right)\zeta(1-z)$$

Hence

$$\Gamma(z)\zeta(z)\sin(\pi z) = (2\pi)^z \sin\left(\frac{\pi z}{2}\right)\zeta(1-z)$$

Using

(40) $$\Gamma(z)\Gamma(1-z)\sin(\pi z) = \pi$$

this becomes

$$\pi\zeta(z) = \Gamma(1-z)(2\pi)^z \sin\left(\frac{\pi z}{2}\right)\zeta(1-z)$$

Using

$$\pi^{-\frac{1}{2}}2^{z-1}\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right) = \Gamma(z)$$

this becomes

$$\pi\zeta(z) = \pi^{-\frac{1}{2}}\Gamma\left(\frac{1-z}{2}\right)\Gamma\left(\frac{2-z}{2}\right)\pi^z \sin\left(\frac{\pi z}{2}\right)\zeta(1-z)$$

Using (40) again this becomes

$$\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z)$$

Calling the left hand side $\xi(z)$ this is a symmetry relation $\xi(z) = \xi(1-z)$.

The symmetry of ξ suggests that the vertical line defined by $\Re(z) = \frac{1}{2}$ plays a special role for ξ. The famous **Riemann hypothesis** is the conjecture that all zeros of ξ are on this vertical line. This is one of teh seven Millennium Prize Problems of the Clay Institute, see http://www.claymath.org .

We shall argue here that ξ has no zeros outside the strip $0 \le \Re(z) \le 1$, and thus get a hint of what this problem has to do with prime numbers. By symmetry, and since the Gamma function has no zeros on the positive half plane, it suffices to show that ζ has no zeros in $\Re(z) > 1$.

We observe that for $\Re(z) > 1$ we have by the unique factorization theorem for integers

$$\sum_{n=1}^{\infty} n^{-z} = \prod_{p \text{ prime}} \sum_{k=0}^{\infty} p^{-kz}$$

(41) $$= \prod_{p \text{ prime}} \frac{1}{1-p^{-z}}$$

This formal caluclation can be easily justified because the product on the right converges:

$$\sum_{p \text{ prime}} \left|\frac{1}{1-p^{-z}} - 1\right| \le \sum_{p \text{ prime}} \left|\frac{p^{-z}}{1-p^{-z}}\right|$$
$$\le \sum_{n=1}^{\infty} \left|\frac{n^{-z}}{1-2^{-z}}\right| < \infty$$

provided $\Re(z) > 1$ Moreover, this calculation also shows that the infinite product (41) does not vanish in $\Re(z) > 1$.

The corona theorem

We dedicate this chapter of the course to Tom Wolff, who tragically died in a car accident earlier this year. Wolff's first famous contribution to Mathematics around 1978 (as graduate student at Berkeley) was a simplification of the proof of the socalled Corona theorem, proved originally by L. Carleson in 1962. We shall discuss the corona theorem and its proof with the simplifications by Tom Wolff.

As a warmup, we consider the ring of complex polynomials of the form

$$\sum_{k=1}^{K} a_k z^k$$

with the usual structure of addition and multiplication. A polynomial f is said to divide a polynomial g if there is a polynomial h such that $gh = f$

We have the following basic lemma

Lemma 30 *Let $f_1, \ldots, f_n$ be non-zero complex polynomials. If for each $z \in \mathbf{C}$ there is one index $1 \leq l \leq n$ such that $f_k(z) \neq 0$, then there exist polynomials $g_1, \ldots, g_n$ such that*

$$\sum_{k=1}^{n} g_k f_k = 1$$

Proof: Fixing $f_1, \ldots, f_n$, we can pick a linear combination

$$r = \sum_{k=1}^{n} f_k g_k \tag{42}$$

such that r is non-zero and has minimal degree. We claim that r divides all of $f_1, \ldots, f_k$. By symmetry it suffices to show that h divides f_1. Consider a linear combination

$$f_1 - hr = s \tag{43}$$

so that s has minimal degree (possibly $s = 0$). Then s has degree strictly less than h, because otherwise

$$s'(z) = s(z) - \frac{\alpha_s}{\alpha_r} z^{n_s - n_r} r(z)$$

where α_s and α_r are the highest non-zero coefficients of s and r and correspond to the powesr n_s and n_r, was a polynomial of the form (43) of smaller degree than s, a contradiction. However, s is again of the form (42), hence s needs to be 0 (r was minimal nonzero). Thus r divides f_1.

Now if z_0 is a root of r, then z_0 is a root of all f_k, which is impossible by hypotheses of the lemma. Hence r has no root and is therefore constant. Since we can normalize r, this proves the lemma.

□

Now let D denote the unit disc $\{z : |Z| < 1\}$ in $\mathbf{C}$ and let $\overline{D}$ denote the closure of D. Let $H(\overline{D})$ denote the set of all functions $f : \overline{D} \leq \mathbf{C}$ which can be extended to be holomorphic on a neighborhood of $\overline{D}$. If $f \in H(\overline{D})$ we write

$$\|f\|_\infty = \sup_{z \in \overline{D}} |f(z)|$$

Then we observe the following corollary of the above lemma.

Corollary 8 *Let $f_1, \ldots, f_n \in H(\overline{D})$ such that for each $z \in \overline{D}$ there exists a index $1 \leq k \leq n$ such that $f_k(z) \neq 0$. Then there exist $g_1, \ldots, g_n \in H(\overline{D})$*

$$\sum_{k=1}^{n} g_k(z) f_k(z) = 1$$

for all $z \in \overline{D}$.

Proof: Each of the f_k has finitely many zeros on the compoact set $\overline{D}$. Let p_k be a polynomial whose zeros (with multiplicities) are exactly those of f_k inside $\overline{D}$, then

$$f_k = p_k h_k$$

where $h_k \in H(\overline{D})$ has no zeros in $\overline{D}$. By the previous lemma we find polynomials q_k such that

$$\sum_{k=1}^{n} p_k q_k = 1$$

Observe that h_k^{-1} is defined and holomorphic in a neighborhood of $\overline{D}$. Also,

$$\sum_{k=1}^{n} f_k(z)(q_k(z) h_k^{-1}(z))$$

for each $z \in \overline{D}$, which proves the corollary. □

The point of the socalled Corona theorem is a size control of the functions g.

Theorem 8 (Corona Theorem) *Let $f_1, \ldots, f_n \in H(\overline{D})$ such that $\|f_j\|_\infty \leq 1$ for all $1 \leq j \leq n$.*

If

$$\sum_{j=1}^{n} |f_j(z)|^2 \geq \delta > 0$$

for all $z \in \overline{D}$, then there exist $g_1, \ldots, g_n$ in $H(\overline{D})$ such that

$$\sum_{j=1}^{n} f_j(z) g_j(z) = 1$$

for all $z \in \overline{D}$ and

$$\|g_j\|_\infty \leq C(\delta, n)$$

for all $1 \leq j \leq n$.

The condition $\|f\|_\infty = 1$ is just a normalization. The point is, that $\|g\|_\infty$ can be controlled solely by the number n of functions and

$$\inf_{z\in\overline{D}} \max_k |f_k(z)| \tag{44}$$

In astronomy, one calls Corona the light phenomena outside the sun that are visible when the moon covers the sun (eclipse). This theorem is called Corona theorem due to the lack of a Corona: the estimate on the functions g_k depends only on a lower bound on the functions f_k inside the disc $\overline{D}$, but not on the behaviour of the functions f_k outside in e neighborhood of the disc.

We remark that there is no reason for the functions from the previous corollary to satisfy the bounds of the corona theorem.

The functions

$$g_k = \frac{\overline{f_k}}{\sum_{j=1}^n |f_j|^2}$$

do satisfy the estimates of the Corona theorem, but are not holomorphic in general.

Littlewood Paley estimates

Before we enter the proof of the Corona theorem, we need to discuss a version Green's theorem and Littlewood Paley estimates.

Let f be twice continuosly differentiable on an open set in $\mathbf{R}^2$. Recall the definition of the Laplace operator:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Using the Wirtinger derivatives

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

we can write

$$\Delta f = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} f$$

In polar coordinates:

$$r = |z|, \quad \theta = \Im(\log z)$$

The Laplace operator can be written

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial \theta^2}$$

Verification of this formula is left as an exercise.

Now we can derive a formula of Green's type: Let u be twice continuously (totally) differentiable in a neighborhood of $\overline{D}$, then

$$\int\int_D \Delta(u) \log\left(\frac{1}{|z|}\right) \, dx \, dy$$

$$= - \int_0^{2\pi} \int_0^1 \Delta(u) \, (\log r) \; r \, dr \, d\theta$$

$$= - \int_0^{2\pi} \int_0^1 \left(\left(\partial_r^2 u\right) r \log r + (\partial_r u) \log r\right) \, dr \, d\theta$$

$$= - \int_0^{2\pi} \int_0^1 \left(- (\partial_r u) \, (1 + \log r) + (\partial_r u) \log r\right) \, dr \, d\theta - \int_0^{2\pi} \left[(\partial_r u) \, r \log r\right]_0^1 \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \partial_r u \, dr \, d\theta$$

$$= \int_0^{2\pi} u(\theta, r) \, d\theta - 2\pi u(0)$$

In case g is a holomorphic function in $H(\overline{D})$ and $u = g\overline{g}$ we observe

$$\Delta u = 4\partial_z \partial_{\overline{z}}(g\overline{g}) = 4\partial_z g \partial_{\overline{z}} \overline{g} = 4|g'|^2$$

Thus

$$4 \int\int_D |g'(x,y)|^2 \log\left(\frac{1}{|z|}\right) \, dx \, dy = \int_0^{2\pi} |g(e^{i\theta})|^2 \, d\theta - 2\pi |g(0)|^2$$

We remark that this special case could also be obtained by power series expansion: assume

$$g(z) = \sum_{k=0}^{\infty} a_k z^k$$

converges in a neighborhood of $\overline{D}$.

Then, for $z = re^{i\theta}$

$$g(z)\overline{g(z)} = \sum_{k,l} a_k \overline{a_l} r^2 e^{i\theta(k-l)}$$

Integrating over θ gives a contribution only if $k = l$. Hence we obtain

$$\int_0^{2\pi} |g(e^{i\theta})|^2 \, d\theta = 2\pi \sum_{k=0}^{\infty} |a_k|^2$$

$$\int\int_D Ontheotherhand, |g'(x,y)|^2 \log\left(\frac{1}{|z|}\right) \, dx \, dy =$$

$$-2\pi \int_0^1 \sum_{k=1}^{\infty} n^2 |a_n|^2 r^{2n-2} (\log r) r \, dr$$

$$= 2\pi \int_0^1 \sum_{k=1}^{\infty} n^2 |a_n|^2 \frac{1}{2n} r^{2n-1} \, dr$$

$$= 2\pi \sum_{k=1}^{\infty} n^2 |a_n|^2 \frac{1}{(2n)^2}$$

which gives the above formula since $g(0) = a_0$.

For $f \in H(\overline{D})$ and $1 \leq p < \infty$ define

$$\|f\|_p = \left(\int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}}$$

Define (as before)

$$\|f\|_\infty = \sup_{\theta \in [0, 2\pi]} |f(e^{i\theta})|$$

Then our basic estimates of Littlewood Paley type are

Lemma 31 (Paraproduct estimates) *Let $f_1, \ldots, f_n$ be in $H(\overline{D})$. Then*

$$\int \int_D |f_1{}' f_2{}' f_3 \ldots f_n| \log\left(\frac{1}{|z|}\right) \, dx \, dy \leq C_n \prod_{k=1}^{n} \|f_k\|_{p_k}$$

whenever $p_k \in \{1, 2, \infty\}$ and $\sum_{k=1}^{n} \frac{1}{p_k} = 1$.

Remark: It is not much harder to see the estimate for any $1 \leq p_k \leq \infty$, but we will not need this more general statement.

Prove: Assume first all the p_k are 2 or ∞. This means, exactly two of them are 2, the others are ∞. We first observe that we may assume $p_1 = 2$ and $p_2 = 2$. If not, say $p_1 = \infty$, then pick a $k > 2$ with $p_k = 2$ and observe

$$|f_1' f_k| = |(f_1 f_k)' - f_k' f_1| \leq |(f_1 f_k)'| + |f_k' f_1|$$

For the first summand we rename $f_1 f_k$ to be the new F_1 and observe

$$\|F_1\|_2 \leq \|f_1\|_\infty \|f_k\|_2$$

For the second summand we interchange the roles of f_1 and f_k. Proceeding likewise if $p_2 = \infty$, we are therefore reduced to $p_1 = p_2 = 2$.

Using the maximum principle to estimate $|f_k(z)|$ for $k > 2$ on the left hand side we may pass to the case $n = 2$.

Now assume $n = 2$. It suffices to prove the estimate for $\|f_1\|_2 = \|f_2\|_2 = 1$, because we may divide by the norms of f_1 and f_2 otherwise. Then the claimed estimate follows from

$$|f_1{}'||f_2{}'| \leq |f_1{}'|^2 + |f_2{}'|^2$$

on the left hand side and Green's formula

It remains to consider the case that we have $p_k = 1$ for some k, and then necessarily $p_j = \infty$ for $j \neq k$.

By adding a small ϵ to f_k if necessary, we may assume f_k has no zero on the boundary of D. Using Blaschke products

$$B_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$$

which have a single zero at α in the disc and modulus 1 on the boundary of the disc, we can write

$$f_k = Bg_k$$

where g has no 0 in the closed disc and B has modulus 1 on the boundary of the disc. Since g has no zero, we can write it as

$$g = e^h = (e^{\frac{h}{2}})^2 = g_1 g_2$$

Hence

$$f = g_1(g_2 B)$$

and we have

$$\|f\|_1 = \|g_1\|_2 \|g_2 B\|_2$$

Thus the estimate follows as an application from the previous case. (Observe that $f' = {g_1}' g_2 B + g_1(g_2 B)'$.) $\square$

Elliptic functions

The function

$$f(z) = e^{2\pi i z}$$

is an example of a periodic holomorphic function

$$f(z) = f(z+1)$$

We shall be interested in doubly periodic functions

$$f(z) = f(z+\omega_1) + f(z+\omega_2) \tag{45}$$

where ω_1 and ω_2 are two non-zero complex numbers which are linearly independent as vectors in the real vector space $\mathbf{R}^2$. Thus the function f satisfies

$$f(z) = f(z+\omega) \tag{46}$$

for all ω in the lattice

$$L = \{n_1\omega_1 + n_2\omega_2, n_1, n_2 \in \mathbf{Z}\}$$

If ω_1 and ω_2 were linerly dependent, then either L would degenerate to

$$\{n\tilde{\omega}, n \in \mathbf{Z}\}$$

for some ω, or it would have an accumulation point and thus not lead to intteresting invariant functions.

Meromorphic functions satisfying (46) are called doubly periodic or elliptic functions.

Lemma 32 *There are no entire functions f satisfying (45) other than the constant functions*

Proof: If A is the interior parallelogramm spanned by the vectors ω_1 and ω_2, then every point of the plane can be written as

$$z + \omega$$

with $z \in \overline{A}$ (closure of A) and $\omega \in L$ integers. Since A is compact, $f(A)$ is compact, and by periodicity $f(\mathbf{R}^2)$ is compact. Hence f is constant by the Liouville theorem. $\square$

Let γ_1 be the straight line from 0 to ω_1, γ_2 the straight line form ω_1 to $\omega_1 + \omega_2$, γ_3 the straight line form $\omega_1 + \omega_2$ to ω_2, and γ_4 the straight line from ω_2 to 0. Let γ be the composition of these four paths, thus γ is a closed curve along the boundary of A.

Lemma 33 *Let f be a non-constant elliptic function which has no poles or zeros on the boundary ∂A. Let $z_1, \ldots, z_n$ be the zeros and poles in A with orders $m_1, \ldots, m_n$ (positive of zero and negative if pole). Then*

$$\sum_{k=1}^{n} m_k = 0 \tag{47}$$

and

$$\sum_{k=1}^{n} m_k z_k = \omega \tag{48}$$

for some $\omega \in L$.

Proof: The sum of orders in the interior of A is given by a constant multiple of the integral

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz$$

$$= \left(\int_{\gamma_1} \frac{f'(z)}{f(z)}\, dz + \int_{\gamma_2} \frac{f'(z)}{f(z)}\, dz + + \int_{\gamma_3} \frac{f'(z)}{f(z)}\, dz + \int_{\gamma_4} \frac{f'(z)}{f(z)}\, dz \right)$$

$$= \left(\int_{\gamma_1} \frac{f'(z)}{f(z)}\, dz + \int_{\gamma_2} \frac{f'(z)}{f(z)}\, dz - \int_{\gamma_1} \frac{f'(z)}{f(z)}\, dz + \int_{\gamma_2} \frac{f'(z)}{f(z)}\, dz \right)$$

$$= 0$$

Here we have used periodicity of f. Thus the sum of orders is zero. The second sum is given by

$$\frac{1}{2\pi i} \int_\gamma z \frac{f'(z)}{f(z)}\, dz$$

$$= \frac{1}{2\pi i} \left(\int_{\gamma_1} z \frac{f'(z)}{f(z)}\, dz + \int_{\gamma_2} z \frac{f'(z)}{f(z)}\, dz + + \int_{\gamma_3} z \frac{f'(z)}{f(z)}\, dz + \int_{\gamma_4} z \frac{f'(z)}{f(z)}\, dz \right)$$

$$= \frac{1}{2\pi i}\left(-\omega_2 \int_{\gamma_1} \frac{f'(z)}{f(z)}\,dz + \omega_1 \int_{\gamma_2} \frac{f'(z)}{f(z)}\,dz\right)$$
$$= n_1\omega_1 + n_2\omega_2$$

Now n_1 is an integer because we can write f on an appropriate convex domain containing the path γ as e^g, hence

$$\int_{\gamma_1} \frac{f'(z)}{f(z)}\,dz = \int_{\gamma_1} g'(z)\,dz = g(\omega_1) - g(0)$$

Since $f(\omega_1) = f(0)$ we conclude that $g(\omega_1) - g(0)$ is an integer multiple of $2\pi i$. Likewise we conclude that n_2 is an integer. □

A non-constant elliptic function has only finitely many poles or zeros in any bounded region. Therefore it is easy to see that we can always find some translate of the parallelogram ∂A which has no zeros or poles, and we can apply a variant of the lemma in this situation.

With the notation as in the lemma call

$$\frac{1}{2}\sum_{k=1}^{n} |m_i|$$

the degree of the elliptic function. Thus the degree is the negative sum of the order of poles of f in the fundamental domain. Translation of an elliptic function deos not change the degree. If the elliptic function has zeros or poles on the boundary of ∂A, then we define the degree to be the degree of an apppropriate translate.

As a corollary we observe that an elliptic function cannot have degree 1. Assume it did. By translation we can assume it to have no zeros or poles on the boundary of A. Then it has one simple zero z_1 and one simple pole z_2 at A. Then

$$z_1 - z_2 = n_1\omega_1 + n_2\omega_2$$

But since z_1 and z_2 are both in the fundamental domain A, this implies $z_1 - z_2 = 0$ which is obviously absurd.

For $m \geq 3$, It is easy to see that the sequence of holomorphic functions on $\mathbf{R}^2 \setminus L$

$$Q_m(z) = \sum_{n_1,n_2=-n}^{n} \frac{1}{(z - n_1\omega_1 + n_2\omega_2)^m}$$

converges uniformly on compact sets to a holomorphic function in $\mathbf{R}^2 \setminus L$ which extends to an elliptic function on $\mathbf{R}^2$ of degree m. It has a pole of order m at 0 and no other non-equivalent poles.

If m is odd, then , Q_m is antisymmetric:

$$Q_m(-z) = -Q_m(z)$$

This implies that

$$Q_m\left(\frac{\omega_1}{2}\right) = \frac{1}{2}\left(Q_m\left(\frac{\omega_1}{2}\right) + Q_m\left(\frac{\omega_1}{2} - \omega_1\right)\right) = 0$$

and likewise

$$Q_m\left(\frac{\omega_2}{2}\right) = Q_m\left(\frac{\omega_1+\omega_2}{2}\right) = 0$$

In the case $m = 3$, these are all zeros of Q_3 (up to equivalent ones). Thus for $\alpha \neq 0$ and β any complex numbers we obtain an elliptic function

$$P(z) = \alpha\frac{Q_5(z)}{Q_3(z)} + \beta$$

which has degree 2 and a pole of order 2 ot the origin. We shall choose α and β so that

$$P(z) = z^{-2} + a_2 z^2 + \text{higher}$$

near $z = 0$. Moreover, P is even because Q_5 and Q_3 are odd. The function P is called Weierstraß P-function.

The derivative P' of P is an elliptic function of degree 3 with pole of order 3 at 0. It is odd because P is even. Thus P'^2 is even. It has a pole of order 6 at 0, therefore it is of the form

$$P'(z) = a_{-6}z^{-6} + a_{-4}z^{-4} + a_{-2}z^{-2} + a_0 + a_2 z^2 + \dots$$

Hence there are complex numbers g_0, g_1, g_2, g_3 such that

$$P'(z)^2 + g_0 P(z)^3 + g_1 P(z)^2 + g_2 P(z) + g_3$$

has the same principal part and constant term as $P'(z)^2$. Thus the difference is a holomorphic elliptic function vanishing at 0, and therefore is constant 0. Indeed, the choice of the coefficients α and β is so that $g_0 = -4$ and $g_1 = 0$. Thus

$$P'(z)^2 = 4P(z)^3 - g_2 P(z) - g_3$$

We can factor the polynomial on the right into its linear factors. The left hand side vanishes if z is $\omega_1/2$, $\omega_2/2$, or $(\omega_1+\omega_2)/2$, by the same reasoning as for Q_3. Thus the right hand side vanishes if $P(z)$ is either one of $e_1 = P(\omega_1/2)$, $e_2 = P(\omega_2/2)$, $e_3 = P((\omega_1+\omega_2)/2)$. Thus

$$P'(z)^2 = 4(P(z) - e_1)(P(z) - e_2)(P(z) - e_3)$$

According to (47) the function P takes every value exactly twice with multiplicity. Since the multiplicity at $\omega_1/2$, $\omega_2/2$ and $(\omega_1+\omega_2)/2$ is two (the derivative of P vanishes), we see that $e - 1$, $e - 2$, and e_3 are all different. The polynomial on the right has no double root.

More on elliptic functions, Birch/Swinnerton-Dyer

As before we fix the lattice L.

That $(P')^2$ can be written as a polynomial in P has a generalization:

Lemma 34 *Every even elliptic function f is a rational function of $P(z)$:*

$$f(z) = \frac{\sum_{k=1}^{n} a_k P(z)^k}{\sum_{k=1}^{m} b_k P(z)^k}$$

Remark: Let f be any elliptic function. We split it as

$$f(z) = g(z) + h(z)$$

where $g(z) = (f(z) + f(-z))/2$ is even and h is odd. Then $h(z)/P'(z)$ is an even elliptic function. Thus we can write f as a sum of a rational function in $P(z)$ and $P'(z)$ times another rational function in $P(z)$.

Proof: By multiplying with a function of the type $(P(z) - (P(z_1))^n$ we can get rid of a pole at $z_1 \notin L$. Thus we can get rid of all finitely many poles in the closure of A except possibly those at the corners. We obtain a new even elliptic function and call it again f. If $2n$ is the (negative) highest order term in the principle part of the Laurent series of f at 0, we can get rid of this term by subtracting an appropriate multiple of $P(z)^n$. Iterating this we can get rid of the entire principal part at the origin. Thus we arrive at a holomorphic elliptic function, which therefore has to be constant. Reversing the previous steps shows that the original function was a rational function in $P(z)$. $\Box$

We shall prove the following explicit formula:

$$P(z) = \frac{1}{z^2} + \lim_{n\to\infty} \sum_{\omega\in L, \omega\neq 0, |\omega|<n} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

Call the right hand side $Q_2(z)$, it can be seen to converge uniformly on compact sets in $\mathbf{R}^2 \setminus \mathbf{Z}^2$ thanks to the extra terms ω^{-2}. Since the sequence converges uniformly on compact sets, the derivative converges pointwise and therefore

$$-\frac{1}{2}{Q_2}'(z) = \frac{1}{z^3} + \lim_{n\to\infty} \sum_{\omega\in L, \omega\neq 0, |\omega|<n} \frac{1}{(z-\omega)^3} = Q_3(z)$$

Since $\frac{Q_3}{P'}$ has no zeros and poles outside L, it must be constant. Thus $(Q_2)' = \alpha P'$ for some constant α and

$$Q_2 = \alpha P + \beta$$

However, the Laurent series of Q_2 is of the form

$$z^{-2} + a_2 z^2 + \dots \quad ,$$

therefore $Q_2 = P$.

The Weierstraß P-function satisfies an addition theorem just like sin and cos.

Lemma 35 *We have for any z_1 and z_2 in $\mathbf{R}^2$:*

$$P(z_1 + z_2) = -P(z_1) - P(z_2) + \frac{1}{4}\left(\frac{P'(z_1) - P'(z_2)}{P(z_1) - P(z_2)}\right)^2 \tag{49}$$

(50) $P'(z_1+z_2) = \left(\frac{P'(z_1)-P'(z_2)}{P(z_1)-P(z_2)}\right) P(z_1+z_2) + \left(\frac{P'(z_1)P(z_2)-P'(z_2)P(z_1)}{P(z_2)-P(z_1)}\right)$

l

Proof: Fix z_1 and z_2 such that $P(z_1) \neq P(z_2)$ and consider coefficients α and β such that

(51) $$P'(z_1) - \alpha P(z_1) - \beta = 0$$

(52) $$P'(z_2) - \alpha P(z_2) - \beta = 0$$

The function

$$P'(z) - \alpha P(z) - \beta$$

has degree 3. Thus it has a third zero z_3 (which might be equal to z_1 or z_2 at this point.) Then we have

$$-3 \times 0 + z_1 + z_2 + z_3 \in L$$

By periodicity we have

$$P(z_3) = P(-z_1 - z_2) = P(z_1 + z_2)$$

$$P'(z_3) = P'(-z_1 - z_2) = -P(z_1 + z_2)$$

Te polynomial

$$(\alpha x + \beta)^2 - 4x^3 + g_2 x + g_3$$

has the thre eroots $P(z_1)$, $P(z_2)$, $P(z_3)$ and can therfore be written as

$$4(x - P(z_1))(x - P(z_2))(x - P(z_3))$$

Comparing the coefficient in front of x^2 gives

$$P(z_1) + P(z_2) + P(z_3) = \frac{\alpha^2}{2}$$

Using (51) and (52) to express α gives (49) Using (51) and (52) to express β gives (50) We have proved (49) and (50) only under the assumption $P(z_1) \neq P(z_2)$. However, fixing z_1, then both sides of the equation coincide for an open set of z_2, hence for all z_2. □

Given L, we call a point z rational if $P(z)$ and $P'(z)$ are rational or if $z \in L$. The previous addition theorems imply that if z_1 and z_2 are rational, then $z_+1 + z_2$ are rational. Thus the rational points (if there are any) form a subgroup of $\mathbf{R}^2$. The rank r of this group is the maximal number of linear independent elements $z_1, \ldots, z_r$ in this group in the sense that

$$\sum_{k=1}^{r} n_k z_k \in L$$

with integers $n_1, \ldots, n_r$ implies $n_k = 0$ for all $k = 1, \ldots, r$.

Now assume that L is a lattice so that g_2 and g_3 are integers. Let

$$\Delta = g_2^3 - 27g_3^2$$

For each prime number p let N_p be the number of pairs (x, y) of residue classes mod p so that

$$y^2 \equiv 4x^3 - g_2 x - g_3 \mod p$$

Then the Birch/Swinnerton-Dyer conjecture says

Conjecture 1 *The function*

$$L(s) = \prod_{gcd(p,2\Delta)=1} \left(1 - (p - N_p)p^{-s} + p^{1-2s}\right)^{-1}$$

defined originally for $\Re(s) > 1$, has a meromorphic extension near $s = 1$ which has a pole of order r at 1 (or rather $-r$ if we use our convention to count orders of poles negative) where r is the rank of the group of rational points for this lattice L.

Remarks: The function LK is a variant of the Riemann zeta function (in the form of its Euler product).

In fact it is known that this function has a meromorphic extension to the entire plane, a result that is closely connected to the recent solution of Fermat's last theorem. So the difficult part is to obtain the order of this meromorphic function at 1.

To conclude the discussion of elliptic functions, we remark that the name elliptic function comes form integration theory. Let $arcP$ denote the inverse function of P in a suitable domain. Then

$$arcP(P(z)) = z$$

in this domain. Taking the derivative gives

$$arcP'(P(z)) = \frac{1}{P'(z)} = \frac{1}{\sqrt{4P(z)^3 - g_2 P(z) - g_3}}$$

Or

$$arcP'(t) = \frac{1}{\sqrt{4t^3 - g_2 t - g_3}}$$

and the primary interest is for t real. Then

$$arcP(y) - arcP(x) = \int_x^y \frac{1}{\sqrt{4t^3 - g_2 t - g_3}} \, dt$$

Integrals of the type of the right hand side have been extensively studied in connection with integrals along a parameter varying on an ellipse. Thus the name elliptic functions, though these connections to elliptic integrals are no longer at the center of the interest.