A Proof of Carleson's theorem

Christoph Thiele

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Carleson's Theorem

For each Schwartz function $f \in \mathcal{S}(\mathbf{R})$ define

$$\mathbf{C}f(x) := \text{sup}_N \left| \int_{-\infty}^{N} \widehat{f}(\xi) e^{2\pi i \xi x} \, d\xi \right| \quad ,$$

where the Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(\xi) := \int f(x) e^{-2\pi i \xi x} \, dx \quad .$$

Then $\mathbf{C}$ is of weak type $2, 2$:

Theorem (Carleson, 1966)

There is a $C > 0$ such that for all $f \in \mathcal{S}(\mathbf{R})$

$$\text{sup}_{\lambda > 0} \, \lambda \, |\{x : \mathbf{C}f(x) > \lambda\}|^{\frac{1}{2}} \leq C \|f\|_2$$

Wave Packets

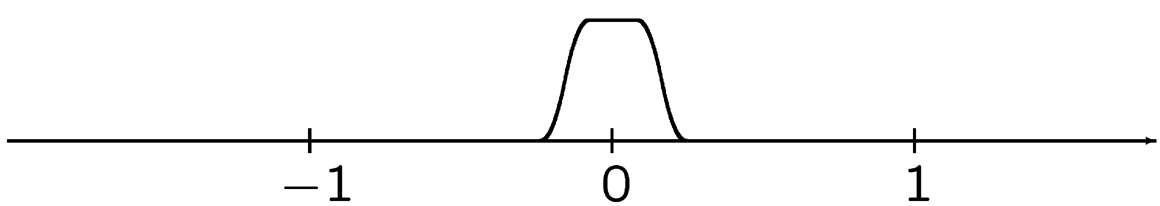
Define translation, modulation, dilation by:

$$T_y f(x) = f(x - y)$$

$$M_\eta f(x) = e^{2\pi i \eta x} f(x)$$

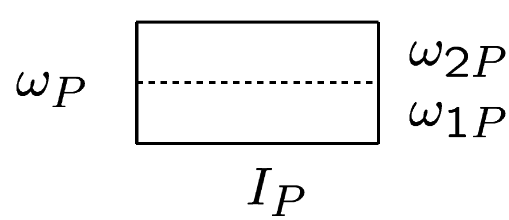
$$D^p_\Lambda f(x) = \Lambda^{-p} f(\Lambda^{-1} x)$$

Pick $\phi \in \mathcal{S}(\mathbf{R})$ so that $\widehat{\phi}$ looks like



Define the wave packet ϕ_{1P} associated to a rectangle $P = I_P \times \omega_P$ of area 1 by

$$\phi_{1P} = M_{c(\omega_{1P})} T_{c(I_P)} D_{|I_P|} \phi$$



The Discrete Hilbert Transform

A *tile* is a rectangle of area 1 of the form

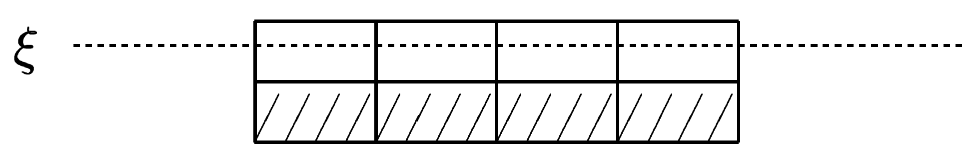
$$[2^k n, 2^k(n+1) \times [2^{-k} l, 2^{-k}(l+1))$$

with integers k, n, l. Let $\overline{\mathbf{P}}$ be the set of tiles.

Define for $\xi \in \mathbf{R}$

$$A_\xi f = \sum_{P \in \overline{\mathbf{P}}} 1_{\omega_{2P}}(\xi) \langle f, \phi_{1P} \rangle \phi_{1P}$$

This is nonzero, positive semidefinite, and vanishes if $\widehat{f}$ is supported in $[\xi, \infty]$.



Observe that for integers k we have

$$A_\xi = D^2_{2^{-k}} A_{2^{-k}\xi} D^2_{2^k}$$

The Hilbert transform as average

Define H_ξ by the average

$$\lim_{N\to\infty} \oint M_{-\eta} T_{-y} D^2_{2^{-\kappa}} A_{2^{-\kappa}(\eta+\xi)} D^2_{2^\kappa} T_y M_\eta$$

Where the average is over the set of all

$$(\eta, y, \kappa) \in [-N, N] \times [-N, N] \times [0, 1]$$

Again this nonzero and vanishes if $\widehat{f}$ is supported in $[\xi, \infty]$.

Moreover it commutes with translations T_z, $z \in \mathbf{R}$ and with dilations $M_\xi D^2_\lambda M_{-\xi}$, $\lambda > 0$ about the point ξ. Hence

$$H_\xi f(x) = c_\xi \int_{-\infty}^{\xi} \widehat{f}(\eta) e^{2\pi i x \eta} \, d\eta$$

Conjugation with M_η gives that c_ξ indeed is independent of ξ.

The Discrete Carleson Theorem

We will prove that

$$\left\| \sup_\xi \left| A_\xi f \right| \right\|_{2,\infty} \leq C \|f\|_2$$

Then by averaging:

$$\left\| \sup_\xi \left| H_\xi f \right| \right\|_{2,\infty} \leq C \|f\|_2$$

The first estimate follows from

$$\sum_{P \in \overline{\mathbf{P}}} \left| \langle f, \phi_{1P} \rangle \langle \phi_{1P} \cdot (1_{\omega_{2P}} \circ N), 1_E \rangle \right| \leq C \|f\|_2 |E|^{\frac{1}{2}}$$

for all $f \in \mathcal{S}(\mathbf{R})$, measurable sets E, and measurable functions N. We can assume $\overline{\mathbf{P}}$ is finite, $\|f\| = 1$, $|E| = 1$. Thus we have to show

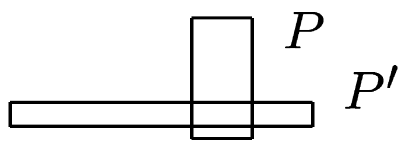
$$\sum_{P \in \overline{\mathbf{P}}} \left| \langle f, \phi_{1P} \rangle \left\langle \phi_{1P}, 1_{E_{2P}} \right\rangle \right| \leq C$$

where

$$E_{2P} = E \cap \{x : N(x) \in \omega_P\}$$

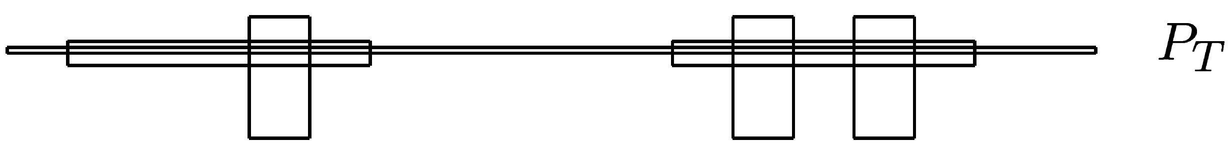
<u>Trees</u>

Define $P < P'$ if $I_P \subset I_{P'}$ and $\omega_{P'} \subset \omega_P$.

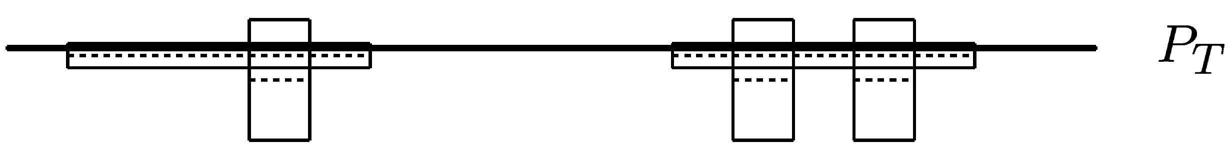


If $P < P'$ we have two possibilities:

A tree T is a set of tiles P such that there exists a tile P_T with $P < P_T$ for all $P \in T$. We do not assume $P_T \in T$.



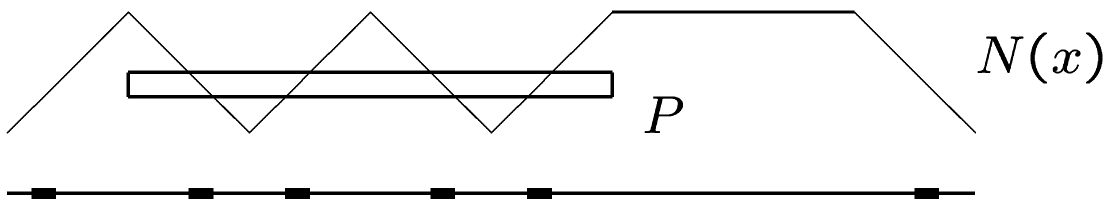
A 2-tree satisfies $\omega_{2P_T} \subset \omega_{2P}$ for all $P \in T$:



Mass

With E the given set of measure 1 define

$$E_P = E \cap \{x : N(x) \in \omega_P\}$$



For $\mathbf{P}$ a set of tiles define

$$\text{mass}(\mathbf{P}) := \max_{P \in \mathbf{P}, P' \in \overline{\mathbf{P}} : P < P'} |I_{P'} \cap E_{P'}| / |I_{P'}|$$

Mass- Proposition

Let $\mathbf{P} \subset \overline{\mathbf{P}}$. Then $\text{mass}(\mathbf{P}) \leq C$. Moreover, $\mathbf{P}$ is the union of two sets $\mathbf{P}_1$ and $\mathbf{P}_2$ with

$$\text{mass}(\mathbf{P}_1) \leq \text{mass}(\mathbf{P})/2$$

and $\mathbf{P}_2$ is the union of trees $T \in \mathbf{T}$ such that

$$\sum_{T \in \mathbf{T}} \left| I_{P_T} \right| \leq C \text{mass}(\mathbf{P})^{-1} \quad .$$

Proof of Mass Proposition

Let $\mathbf{P}_2$ be the set of $P \in \mathbf{P}$ with

$$\text{mass}(\{P\}) \geq \text{mass}(\mathbf{P})/2 \quad .$$

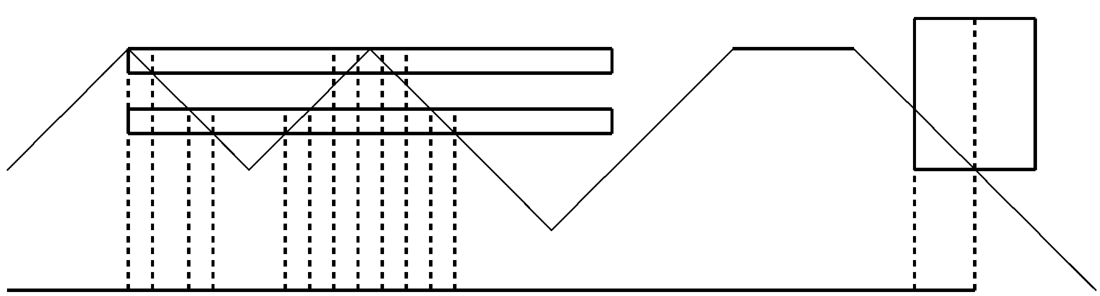
Then $\mathbf{P}_1 = \mathbf{P} \setminus \mathbf{P}_2$ is as required.

Let $\mathbf{P}'$ be the set of maximal tiles in $\overline{\mathbf{P}}$ s.t.:

$$|I_{P'} \cap E_{P'}|/|I_{P'}| \geq \text{mass}(\mathbf{P})/2$$

Then $\mathbf{P}_2$ is a union of trees T with $P_T \in \mathbf{P}'$.

The tiles $P' \in \mathbf{P}'$ are pairwise disjoint.
Hence the sets $I_{P'} \cap E_{P'}$, are pairwise disjoint.



$$\sum_{P' \in \mathbf{P}'} |I_P| \leq \sum_{P' \in \mathbf{P}'} \frac{2|I_{P'} \cap E_{P'}|}{\text{mass}(\mathbf{P})} \leq \frac{2}{\text{mass}(\mathbf{P})}$$

A stronger mass proposition

Define the weight function

$$w(x) = (1 + |x|)^{-100}$$

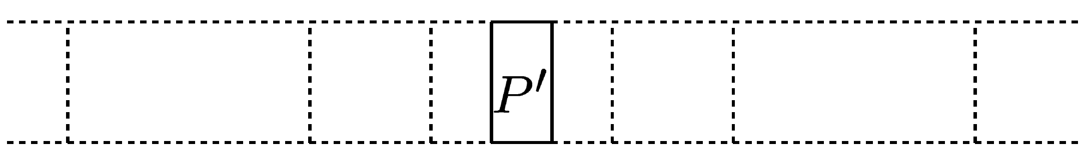
For a tile P define

$$w_P = T_{c(I_P)} D^1_{|I_P|} w$$

Then the mass proposition continues to hold for the modified definition of mass as

$$\max_{P \in \mathbf{P}, P' \in \overline{\mathbf{P}} : P < P'} \int_{E_{P'}} w_{P'}(x)\, dx \ .$$

Proof: Apply the old result to the rectangles



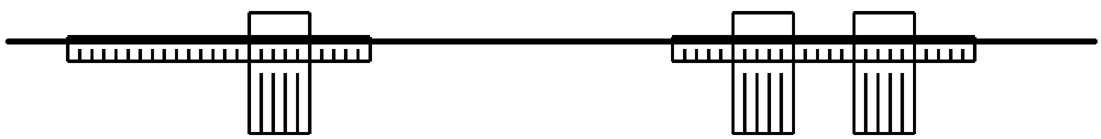
(This page is superfluous in the Walsh model)

Energy

Recall $f \in \mathcal{S}(\mathbf{R})$ with $\|f\|_2 = 1$. Define

$$\text{energy}(\mathbf{P}) := \max_T \left(\left| I_{P_T} \right|^{-1} \sum_{P \in T} |\langle f, \phi_{1P} \rangle|^2 \right)^{\frac{1}{2}}$$

where the max is taken over all 2-trees $T \subset \mathbf{P}$.



P_T

Energy- Proposition

Let $\mathbf{P} \subset \overline{\mathbf{P}}$. Then $\mathbf{P}$ is the union of two sets $\mathbf{P}_1$ and $\mathbf{P}_2$ with

$$\text{energy}(\mathbf{P}_1) \leq \text{energy}(\mathbf{P})/2$$

and $\mathbf{P}_2$ is the union of trees $T \in \mathbf{T}$ such that

$$\sum_{T \in \mathbf{T}} \left| I_{P_T} \right| \leq C \text{energy}(\mathbf{P})^{-2} \quad .$$

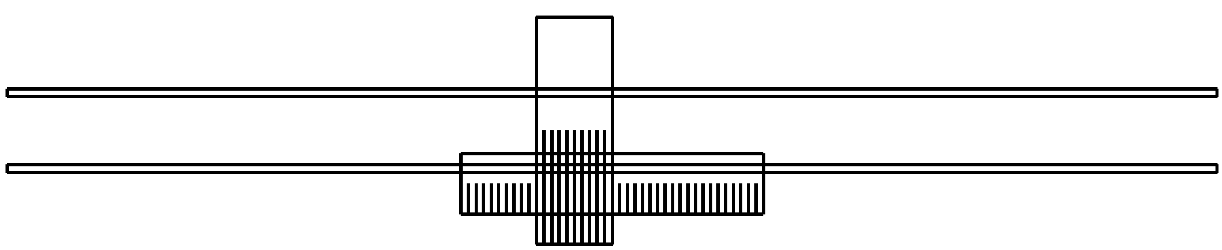
Idea of Energy Proposition

Remove a 2- tree $T \in \mathbf{P}$ such that

$$\left(\left| I_{P_T} \right|^{-1} \sum_{P \in T} |\langle f, \phi_{1P} \rangle|^2 \right)^{\frac{1}{2}} \geq \text{energy}(\mathbf{P})/2 \quad ,$$

the center of ω_{P_T} is minimal (primary goal), and the tree is maximal w.r.t. set inclusion (secondary goal). Iterate until the remainder set $\mathbf{P}_1$ has half the energy of $\mathbf{P}$.

The rectangles $I_P \times \omega_{1P}$ of all $P \in \mathbf{P}_2 = \mathbf{P} \setminus \mathbf{P}_1$ are pairwise disjoint. For assume not, then



If the corresponding ϕ_{1P} were orthogonal:

$$\sum_T |I_{P_T}| \leq 4 \text{ energy}(\mathbf{P})^{-2} \sum_T \sum_{P \in T} |\langle f, \phi_{1P} \rangle|^2$$

$$\leq 4\text{energy}(\mathbf{P})^{-2}$$

End of Proof

Decompose $\overline{\mathbf{P}}$ into sets $\mathbf{P}_n$ with $n \in \mathbf{Z}$ s.t.:

$$\text{mass}(\mathbf{P}_n) \leq \min(C, 2^{2n}) \quad ,$$

$$\text{energy}(\mathbf{P}_n) \leq 2^n \quad ,$$

and $\mathbf{P}_n$ is a union of trees $T \in \mathbf{T}_n$ such that

$$\sum_{T \in \mathbf{T}_n} |I_{P_T}| \leq 2^{-2n} \quad .$$

We will see later that for each tree T:

$$\sum_{P \in \mathbf{T}} \left| \langle f, \phi_{1P} \rangle \left\langle \phi_{1P}, 1_{E_{2P}} \right\rangle \right|$$

$$\leq C \text{energy}(T) \text{mass}(T) \left| I_{P_T} \right| \quad .$$

Hence we have

$$\sum_{n \in \mathbf{Z}} \sum_{T \in \mathbf{T}_n} \sum_{P \in T} \left| \langle f, \phi_{1P} \rangle \left\langle \phi_{1P}, 1_{E_{2P}} \right\rangle \right|$$

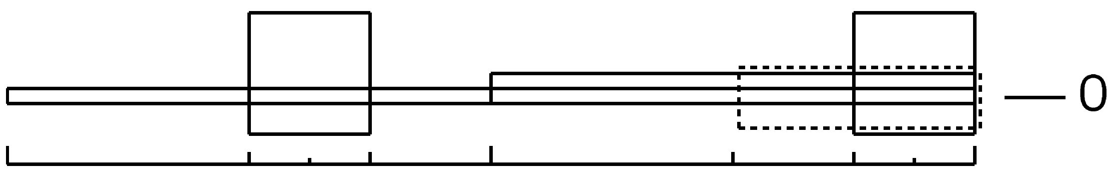
$$\leq C \sum_{n \in \mathbf{Z}} \min(2^n, 2^{-n}) \leq C \quad .$$

This gives Carleson's theorem.

Idea of Tree Estimate

Fix a tree T. Let J be a maximal dyadic interval such that $2J$ does not contain any I_P with $P \in T$. Then

$$J \cap \bigcup_{P \in T : J \subset I_P} E_P \leq C\text{mass}(T)|J| \quad .$$



We have to estimate for certain $|\epsilon_P| = 1$

$$\left| \int \sum_{P \in T} \epsilon_P \langle f, \phi_{1P} \rangle \phi_{1P} 1_{E_{2P}} \right|$$

Pretending ϕ_{1P} is supported in I_P:

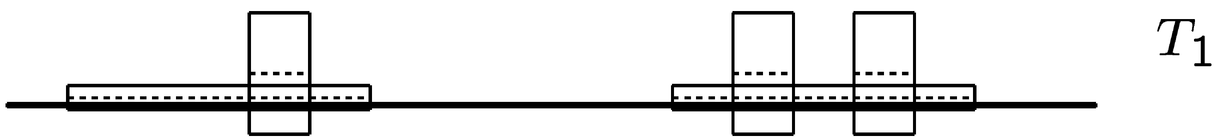
$$\sum\nolimits_J \left\| \sum\nolimits_{P \in T} \epsilon_P \langle f, \phi_{1P} \rangle \phi_{1P} 1_{E_{2P}} \right\|_{L^1(J)} \leq$$

$$C\text{mass}(T) \sum_J |J| \left\| \sum_{P \in T} \epsilon_P \langle f, \phi_{1P} \rangle \phi_{1P} 1_{E_{2P}} \right\|_{L^\infty(J)}$$

The 1-Tree

For $j = 1, 2$ let T_j be the set of $T \in \mathbf{T}$ with such that the center of ω_{P_T} is contained in ω_{jP}.

Consider the tree T_1. The sets $I_P \times \omega_{2P}$ are pairwise disjoint, hence so are the sets E_{2P}.



We pretend that ϕ_{1P} is supported in I_P and hence in I_{P_T} for all $P \in T_1$. Since $\{P\}$ is a 2-tree for each P, we have

$$\sum_J |J| \left\| \sum_{P \in T_1} \epsilon_P \langle f, \phi_{1P} \rangle \phi_{1P} 1_{E_{2P}} \right\|_{L^\infty(J)}$$

$$\leq C \sum_{J \subset I_{P_T}} |J| \text{energy}(T)$$

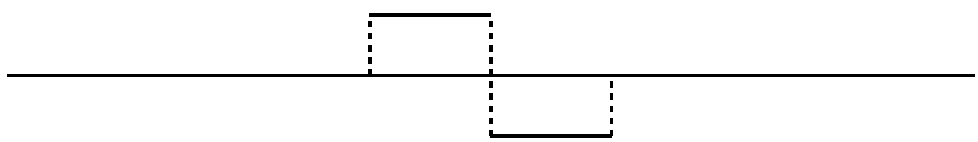
$$\leq C \text{energy}(T) \left| I_{P_T} \right|$$

This gives the tree estimate for T_1.

The 2-Tree

Assume w.l.o.g. that the center of ω_{P_T} is 0.

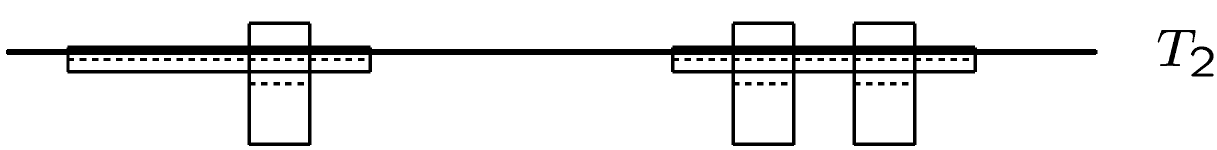
Then ϕ_{1P} has mean 0 for each $P \in T_2$. We pretend that ϕ_{1P} is the Haar function on I_P if $P \in T_2$.



Fix $J \subset I_T$. For each $x \in J$ there are numbers $|J| < |A_x| < |B_x|$ such that for all $P \in T_2$

$$N(x) \in \omega_{2P} \quad \text{iff} \quad A_x \le |I_P| \le B_x \quad ,$$

$$x \in E_{2P} \quad \text{iff} \quad A_x \le |I_P| \le B_x \quad .$$



A Maximal Truncated Singular Integral

We have pointwise on an interval J

$$\left|\sum_{P\in T_2} \epsilon_P \langle f, \phi_{1P}\rangle \phi_{1P} 1_{E_{2P}}\right|$$

$$\leq 2 \max_{A:|J|<A} \left|\sum_{P\in T_2:|I_P|\geq A} \epsilon_P \langle f, \phi_{1P}\rangle \phi_{1P}\right|$$

$$\leq 2 \max_{I:J\subset I} \frac{1}{|I|} \left|\int_I \sum_{P\in T_2} \epsilon_P \langle f, \phi_{1P}\rangle \phi_{1P}\right|$$

With the Hardy-Littlewood maximal function M:

$$\sum_J |J| \left\|\sum_{P\in T_2} \epsilon_P \langle f, \phi_{1P}\rangle \phi_{1P} 1_{E_{2P}}\right\|_{L^\infty(J)}$$

$$\leq C \left\| M\left(\sum_{P\in T_2} \epsilon_P \langle f, \phi_{1P}\rangle \phi_{1P}\right)\right\|_{L^1(I_{P_T})}$$

$$\leq C|I_{P_T}|^{\frac{1}{2}} \left\|\sum_{P\in T_2} \epsilon_P \langle f, \phi_{1P}\rangle \phi_{1P}\right\|_2$$

$$\leq C|I_{P_T}|\text{energy}(T) \quad .$$

Proof of Energy Proposition

We have to prove Bessel's inequality despite leakage of ϕ_{1P} in spatial direction. This is done by a variant of Schur's test.

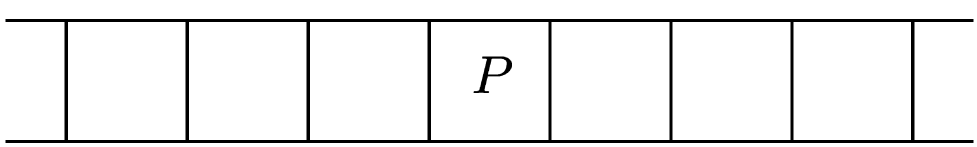
$$\left(\sum_{P\in\mathbf{P}_2} |\langle f, \phi_{1P}\rangle|^2\right)^2 \leq \left\|\sum_{P} \langle f, \phi_{1P}\rangle \, \phi_{1P}\right\|_2^2$$

$$\leq \sum_{P,P':\omega_P=\omega_{P'}} \langle f, \phi_{1P}\rangle \langle \phi_{1P}, \phi_{1P'}\rangle \langle \phi_{1P'}, f\rangle$$

$$+2 \sum_{P,P':\omega_P\subset\omega_{1P'}} \langle f, \phi_{1P}\rangle \langle \phi_{1P}, \phi_{1P'}\rangle \langle \phi_{1P'}, f\rangle$$

$$\leq 2\sum_{P} |\langle f, \phi_{1P}\rangle|^2 \max_P \sum_{P':\omega_P=\omega_{P'}} |\langle \phi_{1P}, \phi_{1P'}\rangle|$$

$$+2\sum_{T}\sum_{P\in T} \langle f, \phi_{1P}\rangle \sum_{P':\omega_P\subset\omega_{1P'}} \langle \phi_{1P}, \phi_{1P'}\rangle \langle \phi_{1P'}, f\rangle$$



The Off-Diagonal Term

$$\text{off.diag.term} \quad \leq 2 \sum_T \left(\sum_{P \in T} |\langle f, \phi_{1P} \rangle|^2 \right)^{\frac{1}{2}} \times$$

$$\left(\sum_{P \in T} \left| \sum_{P' : \omega_P \subset \omega_{1P'}} \langle \phi_{1P}, \phi_{1P'} \rangle \langle \phi_{1P'}, f \rangle \right|^2 \right)^{\frac{1}{2}}$$

But each $\{\mathbf{P'}\}$ is a 2-tree by itself and thus

$$|\langle \phi_{1P'}, f \rangle| \leq |I_{P'}|^{\frac{1}{2}} \text{energy} \mathbf{P}$$

Thus by selection of T:

$$\text{off.diag.term} \quad \leq 2 \sum_T \left(\sum_{P \in T} |\langle f, \phi_{1P} \rangle|^2 \right) \times$$

$$\left(\sum_{P \in T} \left| \sum_{P' : \omega_P \subset \omega_{1P'}} |\langle \phi_{1P}, \phi_{1P'} \rangle| \, |I_{P'}|^{\frac{1}{2}} \left| I_{P_T} \right|^{-\frac{1}{2}} \right|^2 \right)^{\frac{1}{2}}$$