Multilinear singular integrals

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1) General Overview
2) Carleson's theorem
3) 1D Schrödinger operators

Multilinear singular integrals

Theory of L^p estimates for operations with modulation symmetries. So far all on $\mathbf{R}^1$.

Modulation:

$$M_\xi f(x) := f(x)e^{2\pi i \xi x}$$

Example 1:

$$B(f,g)(x) = \int f(x-t)g(x+t)\frac{dt}{t}$$

$$B(M_\xi f, M_\xi g) = M_{2\xi} B(f,g)$$

Example 2:

$$C(f)(x) = \sup_\xi \left| \int_{-\infty}^{\xi} \widehat{f}(\xi) e^{2\pi i \xi x}\, d\xi \right|$$

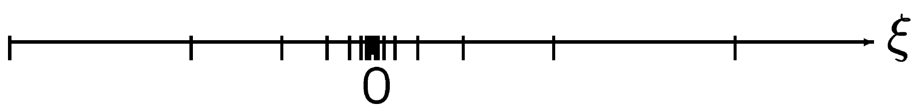
$$C(M_\xi f) = C(f)$$

Example 3: Sometimes operations do not have symmetries themselves, but belong to a class of operations with symmetries.
Example 4: Operations which locally (in Fourier space) have such (approximate) symmetries.

Failure of genuine Littlewood Paley

Littlewood Paley theory = approach of simplifying operations by decomposing

$$f = \sum_{k \in \mathbf{Z}} \phi_k * f \ , \quad \phi_k(x) = 2^k \phi(2^k x)$$



Trouble: Conflict with modulation symmetry!

Example: Carleson's operator. Assume we could estimate each single piece of the Littlewood Paley decomposition. Assume w.l.o.g. that $\widehat{f}$ is compactly supported. Then for some large ξ and k

$$C(f) = C(M_\xi f) = C(\phi_k * M_\xi f)$$

Thus we can estimate Carleson's operator. Individual pieces of the Littlewood Paley decomposition are as hard as the whole problem.

!!!! Wavelets fail for the same reason !!!!

Walsh models

A function f is (over-) determined by

$$\left(\left\langle f, \phi_{k,l,n} \right\rangle\right)_{k,n,l \in \mathbf{Z}}$$

where $\phi \in \mathcal{S}(\mathbf{R})$ is appropriate and

$$\phi_{k,l,n}(x) = 2^{\frac{k}{2}}\ \phi(2^k x - n)\ e^{2\pi i 2^k x l}$$

The multilinear singular integrals under consideration factor NATURALLY over the space of all sequences parametrized by k, n, l.

This allows one to write down NATURAL models by considering

$$\left(\left\langle f, w_{k,n,l} \right\rangle\right)_{k,n,l \in \mathbf{Z}, n,l \geq 0}$$

instead, where $\psi = 1_{[0,1)}$ and

$$w_{k,n,l}(x) = 2^{\frac{k}{2}}\ \psi(2^k x - n)\ \chi_l(2^k x)$$

and χ_l is the l-th character of $\mathbf{R}^+$ with the structure of bitwise addition. These Walsh-models are easier (perfect time-frequency localization), and exhibit the NATURAL structure of the multilinear singular integrals.

Results on multilinear singular integrals

1) Carleson's theorem
Carleson 1966; Fefferman 1972; Lacey - T. 2000; Walsh later: Billard 1967

2a) Bilinear Hilbert transform, locally L^2,
Lacey - T. 1997; Walsh first: T., Thesis 1995

2b) Bilinear Hilbert transform,
Lacey - T. 1999; Walsh later (in 3))

3) Bi- Maximal operator,
Lacey, 2000; Walsh later: T., to appear

4a) Uniform estimates, L^2
T. , Habilitation 1998; Walsh later: T. preprint

4b) Uniform estimates, locally L^2
Grafakos - Li, preprint 2000; no Walsh yet

4c) Uniform estimates,
Li, preprint 2000; no Walsh yet

4d) Parabola- BHT,
Muscalu, thesis 2000; no Walsh

5a) Multilinear generalization, type 1,
Muscalu - Tao - T., preprint 1999 no Walsh yet

5b) Multilinear uniform estimates
Muscalu - Tao -T., work in progress

6a) Multilinear generalization, type 2, L^2:
Muscalu - Tao - T., work in progress, Walsh preprint

6b) Trilinear case all L^p:
Muscalu - Tao - T., work in progress, Walsh preprint

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6z) Results on 1D Schrödinger operators (goal)

Multilinear singular integrals (type 1)

Consider multilinear forms

$$\Lambda : \mathcal{S}(\mathbf{R}) \times \ldots \times \mathcal{S}(\mathbf{R}) \times S(\mathbf{R}) \to \mathbf{C}$$

These correspond to multilinear operators T

$$T : \mathcal{S}(\mathbf{R}) \times \ldots \times \mathcal{S}(\mathbf{R}) \to \mathcal{S}'(\mathbf{R})$$

via duality:

$$\Lambda(f_1, \ldots, f_n) = \int T(f_1, \ldots, f_{n-1})(x) f_n(x)\, dx$$

Observe that the symmetric group on n elements acts on the space of n-linear forms.

By the Schwartz kernel theorem

$$\Lambda(f_1, \ldots, f_n) = \int m(\xi) \prod_{i=1}^{n} \widehat{f}_i(\xi_i)\, d\xi_i$$

for some distribution $m \in \mathcal{S}'(\mathbf{R}^n)$.

Translation invariance

We make the assumption that our multilinear forms are invariant under simultaneous translation of all f_i.

Translation:

$$T_y f(x) = f(x - y)$$

Invariance:

$$\Lambda(T_y f_1, \ldots, T_y f_n) = \Lambda(f_1, \ldots, f_n)$$

This implies for the distribution m:

$$m(\xi) = m(\xi) e^{2\pi i y(\xi_1 + \ldots + \xi_n)}$$

Hence m is a distribution on the hyperplane

$$\Gamma := \{\xi : \xi_1 + \ldots + \xi_n = 0\}$$

In the case of bilinear forms and linear operators this m is the usual multiplier.

Modulation symmetry

Next, we demand some modulation symmetry on multilinear forms.

We choose a subspace

$$\Gamma' \subset \Gamma$$

and demand the modulation symmetry

$$m = T_\eta m$$

for all $\eta \in \Gamma'$.

More generally, one can demand symbol estimates

$$|\partial^\alpha m(\xi)| \leq C_\alpha \mathrm{dist}(\xi, \Gamma')^{-|\alpha|}$$

for all partial derivatives ∂^α tangential to Γ.

Then Λ itself has not necessarily modulation symmetry, but it belongs to a class of forms which has this symmetry.

Theorem (Muscalu, Tao, T)

Let k, n be integers with

$$0 \leq k < n/2$$

Let Γ be the hyperplane in $\mathbf{R}^n$ as before and let Γ' be a k-dimensional subspace of Γ such that the orthogonal projection of Γ' onto any k dimensional space spanned by coordinate axes is nondegenerate.

Let m be a distribution on Γ such that

$$|\partial^\alpha m(\xi)| \leq \mathrm{dist}(\xi, \Gamma')^{-|\alpha|}$$

for all partial derivatives tangential to Γ up to some finite order.

Then the corresponding Λ satisfies

$$|\Lambda(f_1, \ldots, f_n)| \leq C \prod_{i=1}^{n} \|f_i\|_{p_i}$$

as long as

$$\sum_{i=1}^{n} \frac{1}{p_i} = 1 \ , \quad 1 < p_1, \ldots, p_n < \infty$$

Remarks

1) For $k = 0$ there is no modulation symmetry. The result can be derived using Littlewood Paley theory (Paraproducts; Coifman-Meyer)

2) For $k = 1$, $n = 3$ this is due to Gilbert-Nahmod. An important special case is given by the work of Lacey-T.

4) The constraint $k < n/2$ comes from the method of proof. Relaxing this has a flavor similar to improving Kakeya beyond Wolff's results. The case $k = n - 2$ is the multilinear Hilbert transform, we can do only $n \leq 3$.

3) The range of exponents can be extended. The multilinear operator T satisfies

$$L^{1+} \times \dots L^{1+} \times L^2 \dots \times L^2 \to L^{1/(n-k-1/2)+}$$

with $n - 2k$ copies of L^{1+}. The lower bound $1/(n - k - 1/2)$ has a similar reason as $k < n/2$

5) The proof gives constants which depend on how close Γ' is to be degenerate. In some cases this can be improved to uniform bounds.

The constraint $k < n/2$

Assume m is constant in direction Γ' and is smooth and compactly supported in Γ/Γ':

$$|\Lambda(f_1, \ldots, f_n)| = \left| \int_\Gamma m(\xi) \prod_{i=1}^n \widehat{f}(\xi_i)\, d\xi_i \right|$$

$$\leq C \prod_{i=1}^n \|\widehat{f_i}\|_{q_i}$$

whenever

$$k = \sum_{i=1}^n \frac{1}{q_i}$$

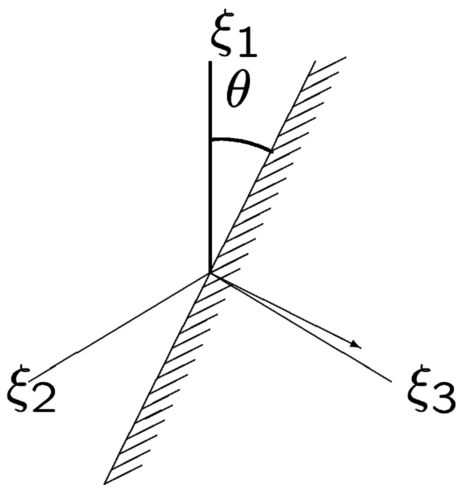
Namely, estimate $n-k-1$ functions f_i in L^∞ and k functions in L^1 and interpolate with the symmetric estimates. We can control $\|\widehat{f_i}\|_{q_i}$ only if $q_i \geq 2$. Assuming $q_i \geq 2$ for all i gives

$$k \leq \sum_{i=1}^n \frac{1}{2} = \frac{n}{2}$$

The obstruction to improve on $k \leq n/2$ comes from the use of Fourier methods. The case $k = n/2$ is on the edge, our argument barely fails (e.g. trilinear Hilbert transform).

The bilinear Hilbert transform

Let $n = 3$ and $k = 1$. Let m be constant on either side of Γ'.



The corresponding bilinear operators form a one parameter family, which is essentially given by

$$H_\alpha(f,g)(x) := p.v. \int f(x-t)g(x-\alpha t)\frac{dt}{t} \quad .$$

Observe the symmetry of this family under permutation of f_1, f_2, f_3. There are three degenerate cases:

$$H_0(f,g) = H(f) \cdot g \ , \quad H_1(f,g) = H(f \cdot g) \ ,$$

$$H_\infty(f,g) = f \cdot H(g) \ .$$

Boundedness of the BHT

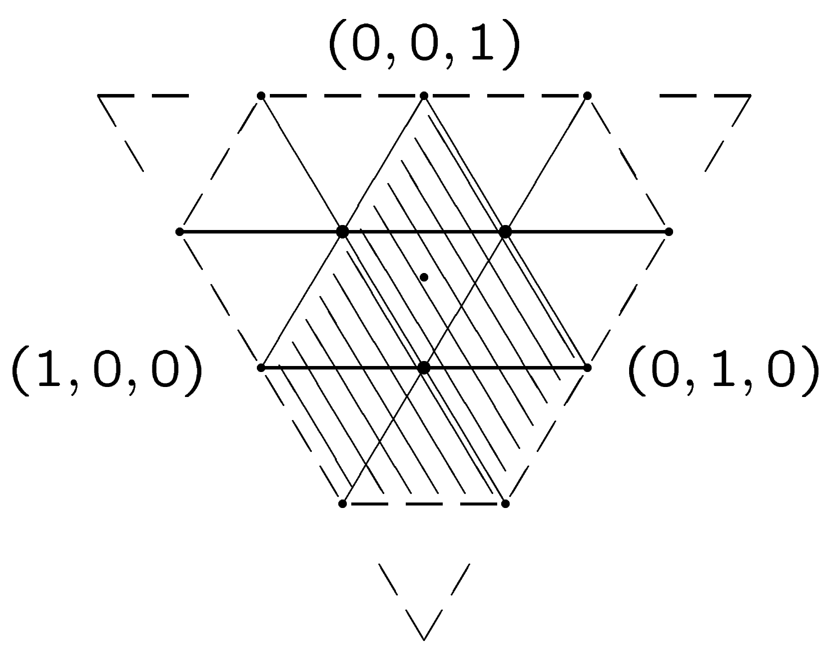
In the non-degenerate case the theory (here due to Lacey-T.) gives

$$\|H_\alpha(f_1, f_2)\|_{p_3{}'} \leq C\|f_1\|_{p_1}\|f_2\|_{p_2}$$

whenever

$$\sum_{i=1}^{3} \frac{1}{p_i} = 1 \ , \quad 1 < p_1, p_2 \leq \infty \ , \ 2/3 < p_3{}' < \infty$$

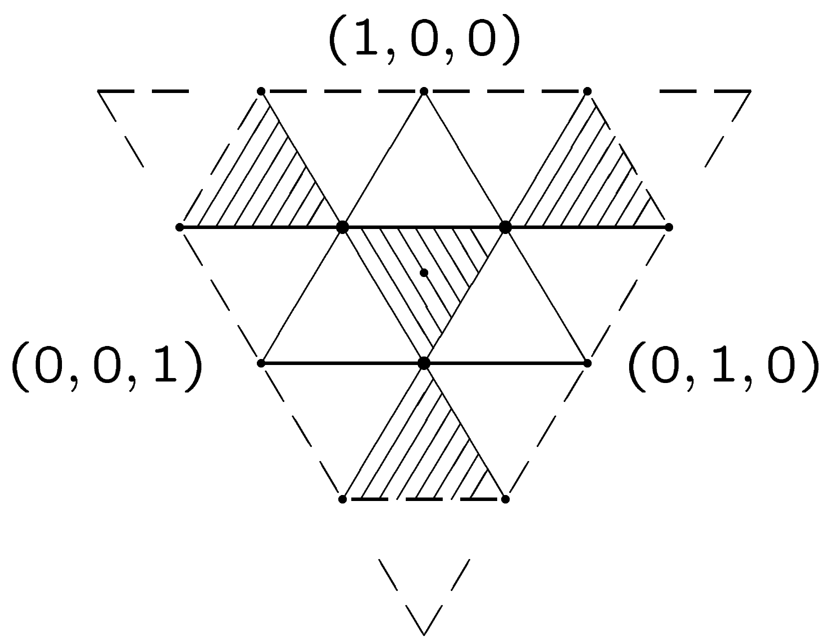
This describes the set of tuples $(1/p_1, 1/p_2, 1/p_3)$ in the following region



Observe that $p_3{}' < 1$ means $p_3 < 0$. The region where $1 < p_i < \infty$ for all i is the convex hull of the indicated points.

Dual estimates

Permuting the indices $1, 2, 3$ gives dual estimates, which give, if all entered in the same diagram, the convex hull of the shaded regions:



Only the shaded regions are proven directly. In the inner triangle all exponents satisfy $2 < p_i < \infty$, so all functions are locally L^2. (First paper Lacey-T.)

The convex hull is then obtained by interpolation.

Estimates in the degenerate case

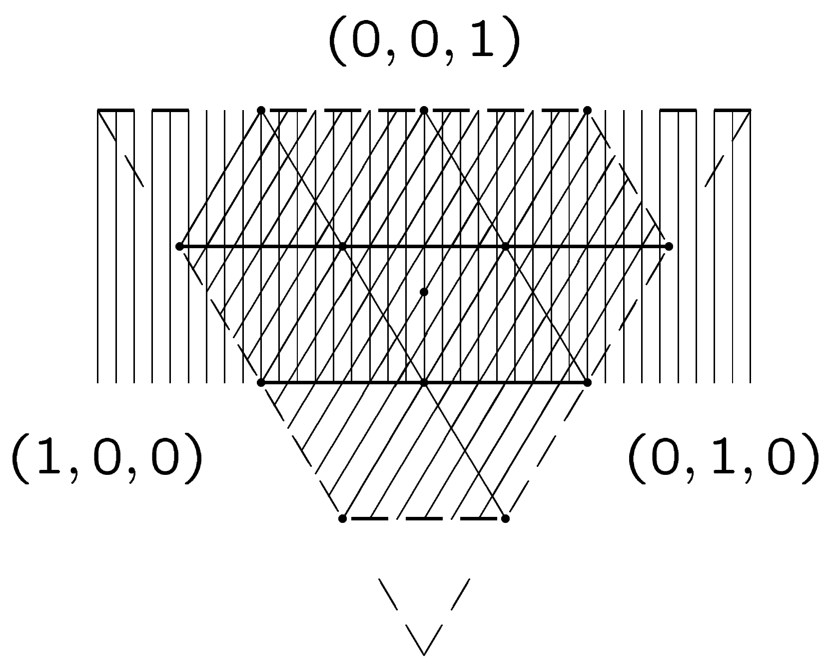
One of the degenerate cases is given by

$$H_1(f_1, f_2) = H(f_1 \cdot f_2)$$

or its dual operators

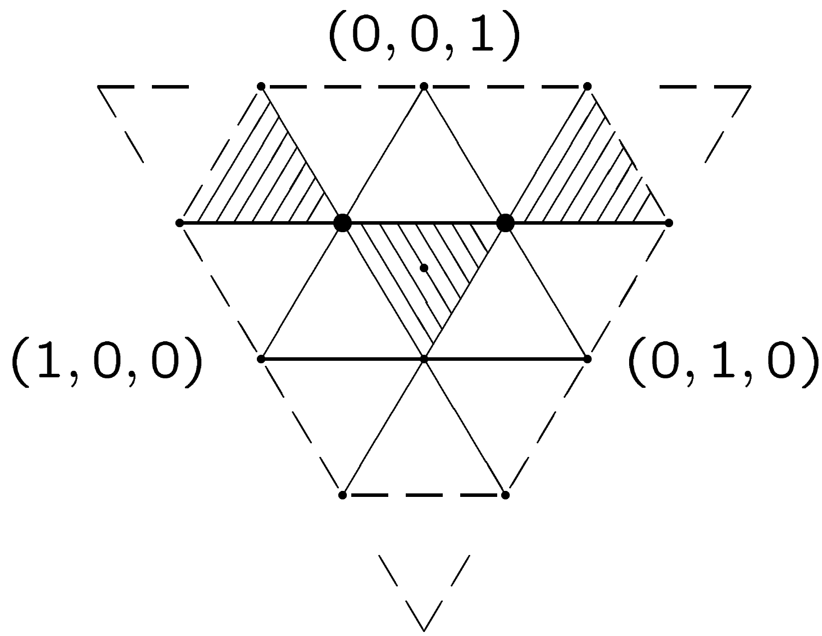
$$f_2 \cdot H(f_3) \ , \quad f_1 \cdot H(f_3)$$

Besides the usual homogeneity, the only constraint for these operators to be bounded is $1 < p_3 < \infty$.



Thus one expects uniform estimates as α approaches 1 to hold in the intersection of the shaded regions.

Uniform estimates



1) T., Habilitation 1998: The two large circles, only weak type

$$L^2 \times L^2 \to L^{1,\infty}$$

2) Grafakos-Li, 2000: Locally L^2 triangle.

3) Li, 2000: the other two shaded triangles.

Observe that the convex hull of these regions still misses some part near $(1,0,0)$ and $(0,1,0)$. This is an open problem.

Motivation for uniform estimates:

Calderon's first commutator is defined as

$$T = [A, M] = AM - MA \quad ,$$

where A is multiplication by a function A and M is convolution with $1/x^2$, i.e.,

$$T(f)(x) := p.v. \int_{\mathbf{R}} \frac{A(x) - A(y)}{(x-y)^2} f(y)\, dy \quad .$$

We are interest in a bound for T in L^2 provided A is Lipschitz. Rewrite $Tf(x)$ as

$$p.v. \int_{\mathbf{R}} \left(\int_0^1 A'(\alpha y + (1-\alpha)x)\, d\alpha \right) \frac{f(y)}{x-y}\, dy$$

$$= p.v. \int_{\mathbf{R}} \left(\int_0^1 A'(x - \alpha t)\, d\alpha \right) f(x-t)\, \frac{dt}{t}$$

$$= \int_0^1 \left(p.v \int_{\mathbf{R}} f(x-t) A'(x-\alpha t)\, \frac{dt}{t} \right) \; d\alpha \quad .$$

Hence we are looking for a uniform bound $H_\alpha : L^2 \times L^\infty \to L^2$ as α varies in $(0,1)$. (Indeed, integrable bounds suffice. E.g. by interpolation methods the T.- Habilitation result suffices to give integrability)

Motivation for Calderon-commutator

Consider the Cauchy integral on a curve γ

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z} \quad .$$

Let γ be a Lipshitz graph $\gamma(x) = x + iA(x)$ and evaluate the Cauchy integral formally on the graph. This gives up to a factor $2\pi i$:

$$p.v. \int_{\mathbf{R}} \frac{f(y)[1 + iA'(y)]\,dy}{x - y + i(A(x) - A(y))} \quad .$$

The factor $1 + iA'(y)$ is inessential for boundedness in L^2. Consider the expansion

$$\frac{1}{x - y + i(A(x) - A(y))}$$

$$= \frac{1}{x-y}\sum_{k=0}^{\infty}(-i)^k \left[\frac{A(x) - A(y)}{x-y}\right]^k \quad .$$

Then the operator corresponding to the first two terms in this expansion is

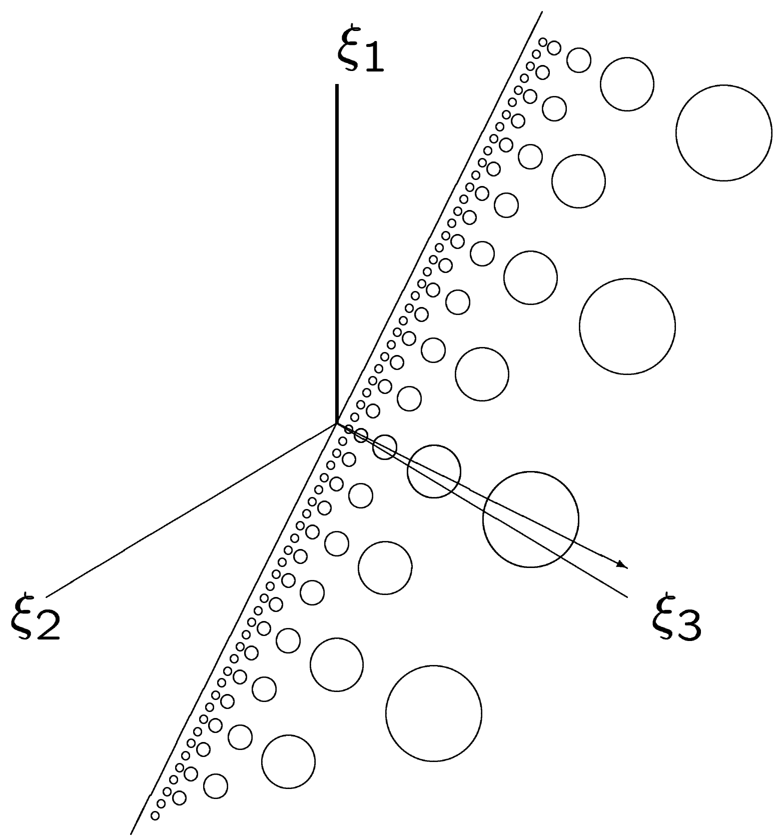
$$H - i[A, M] \quad .$$

The Whitney Decomposition

Assume we are away from the degenerate case. The condition

$$|\partial^\alpha m(\xi)| \leq C_\alpha \mathrm{dist}(\xi, \Gamma')^{-|\alpha|}$$

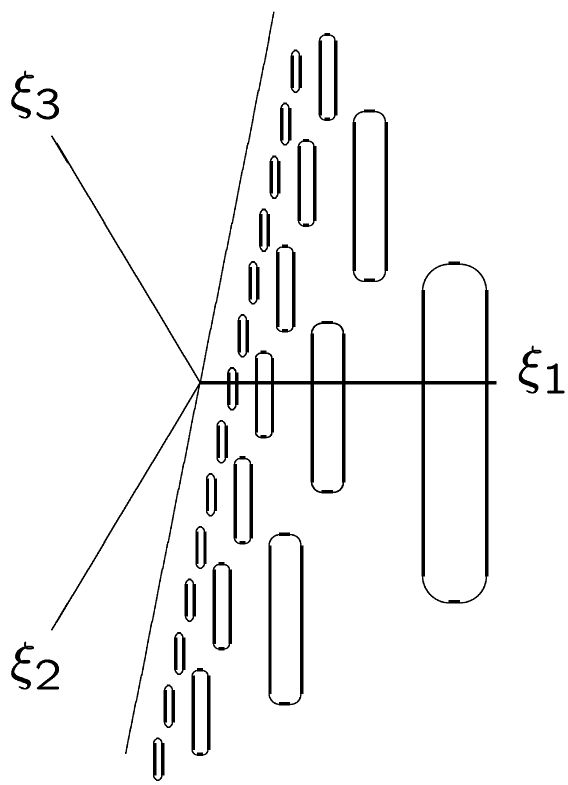
implies that m is essentially constant on the Whitney-regions



At a single scale these regions are not allowed to project onto the same interval when projected onto a coordinate axis. This however happens in the three degenerate cases.

Near the Degenerate Case

In order to prove uniform bounds, one has to formulate the multiplier condition so as to give essentially constant multiplier on adapted regions



This picture has a generalization in the case $k = 1$, $n > 3$. For $k > 1$ the analog of this picture is not known, thus uniform estimates are not understood. In fact even the degenerate cases themselves are not fully understood.

Multilinear uniform estimates

(Work in progress) Theorem: Let $n \geq 3$ and $\Gamma \subset \mathbf{R}^n$ as before. Let Γ' be a subspace of Γ spanned by $(\gamma_1, \ldots, \gamma_n) \in \mathbf{R}^n$. Assume $\gamma_i \neq 0$ for all i. Define distances

$$d(x, y) = \mathrm{sup}_i |x_i - y_i| / |\gamma_i|$$

$$d(x, \Gamma') = \mathrm{inf}_{y \in \Gamma'} d(x, y)$$

Assume all derivatives of m up to some finite order satisfy

$$|\partial^\alpha m(\xi)| \leq \prod_{i=1}^{n} (|\gamma_i| d(x, \Gamma'))^{-|\alpha_i|}$$

(if, say, m is extended to be constant perpendicular to Γ)
Then, uniformly in the choice of Γ',

$$\Lambda(f_1, \ldots, f_n) \leq C \prod_{i=1}^{n} \|f_i\|_{p_i}$$

as long as (all locally L^2 case):

$$\sum_{i=1}^{n} \frac{1}{p_i} = 1 \ , \quad 2 < p_i < \infty$$