Bootcamp

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Summer 2012

Axioms of set theory

The naive idea of sets is that they colelct objects together. Suppose we have a collection of objects

$$a, b, c, d, e, \ldots$$

Then a set could be

$$\{a, c, e\}$$

collecting the objects a, c, and e into one set, that we may call A. Sets can express concepts, for exampelk imagine we have a box of apples, some red and some green. In some ways the set of red apples in the basket expresses the concept of "red". Many mathematical concepts can be expressed by means of sets, and thus purpose of set theory is to formalize the theory of sets.

Rather than basing the theory of sets on putting objects into brackets, one introduces a containment symbol $\in$. If one calls the above set A, then we have for example $a \in A$. We also have $c \in A$ and we have $e \in A$. These are elementary statements, and they are true. The statement $b \in A$ is false.

We are then lead to a theory of statements. A statement of the form $x \in X$ where x is an object and X is a set is called an eleemntary statement, and it is either true or false. There are some formal ways to produce other statements One is using the symbol $\neg$ ("not"), which turns a true statement in a false statement and vice versa. For example $\neg(a \in A)$ is a wrong statement while $\neg(b \in A)$ is a true statement. Two statements can be connected into a new statement by the symbol $\wedge$.

$$(x \in X) \wedge (y \in Y)$$

is true is both $x \in X$ and $y \in Y$ are true, and the combined statement is true if $x(\in X$ is false or $y \in Y$ is false or both $x \in X$, $y \in Y$ are false.

Using $\neg$ and $\wedge$ one can define further interesting symbols, for example

Definition 1

$$(x \in X) \Rightarrow (y \in Y)$$

is the same as the statement

$$\neg((x \in X) \wedge (\neg(y \in Y)))$$

Definition 2
$$(x \in X) \Leftrightarrow (y \in Y)$$
is defined to be the statement
$$((x \in X) \Rightarrow (y \in Y)) \wedge ((x \in X) \Rightarrow (y \in Y))$$

Definition 3
$$(x \in X) \vee (y \in Y)$$
is defined to be the statement
$$\neg((\neg(x \in X)) \wedge (\neg(y \in Y)))$$

Exercise 1 *In all the following exercise you may use the $\neg$ symbol freely, but other than that only the symbols that are stated.*

1. *Express the symbol $\Rightarrow$ in terms of the symbol $\vee$*
2. *Express the symbol $\wedge$ in terms of the symbol $\Rightarrow$*
3. *Prove or disprove: $\wedge$ may be expressed in terms of the symbol $\Leftrightarrow$*

Conceptually, we may restrict attention to $\neg$ and $\wedge$ and treat the other symbols introduced above as derived symbols expressable through $\neg$ and $\wedge$.

Besides $\in$ there is an other symbol creating elementary statement, it is the equality sign $=$. At this point we only allow the equality sign to connect two objects. It returens the value true if the two objects connected are the same, e.g. $a = a$ and $b = b$ are true, while $a = b$ is not true. This requires some a priori agreement of which objects are the same, in the above list our presumption is that each letter stands for its own individual object that is unequal to the objects represented by other letters.

One further way of making a statement is a quantifier $\forall$. It requires a variable x, which is not an object per see but stands for an arbitrary object, The statement
$$\forall x(\neg(x \in A))$$
is true if the statement $(\neg(x \in A))$ is true for all objects x. The expression $(\neg(x \in A))$ is not a statement per se, since x is a variable and not an object. It is called an expression in one free variable. Siuch exporession only becomes a statement in conjcuntion with the quantifyer $\forall x$. The statement of the abopve display for example is then false, becasue the statement $\neg(a \in A)$ is false.

There is again a derived companion quantifyer

Definition 4 *If $P(x)$ is an expression in one free object variable x, then the statement*
$$\exists x(P(x))$$
is defined to be the statement
$$\neg(\forall x(\neg(P(x))))$$

Now the following is a true statement about the above set A

$$\forall x((x \in A) \Leftrightarrow (((x = a) \vee (x = c)) \vee (x = e)))$$

and truth of this statement evidently describes the set A.

One particular set is the empty set $\emptyset$, which contains no object. The following statement is true:

$$\forall x(\neg(x \in \emptyset))$$

Note sets are designed to describe concepts. We would like to describe concepts about sets, so we would like to form sets of sets. For example

$$\{\{a, c\}, \{a, b\}, \{b, c\}\}$$

is a set containing three sets. Informally, it describes the concept of two element sets fromed from the elements a,, b, c. Even without using the list of given objects, one can form many sets. We have already seen $\emptyset$, but one can also form the set $\{\emptyset\}$, which contains one element, namely the emptyset. This set is not equal to the emptys set, since it contains one element while the emptyset contains zero elements. Another set on can form is the set $\{\emptyset, \{\emptyset\}\}$. This set contains two elements. It turns out the sets one can form in this manner are rich enopugh to describe all concepts in mathematics. For example one may denopte the above sets by $0, 1, 2$ and then define 3 to be $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ and so on, and one can start talking about natural numbers without having to have natural numbers as prescribed objects.

To streamline the theory and get rid of difficulties such as having to define equality of objects in some external way, one abandons the list of objects and only considers sets. Note thatw e have to revise a bit of the above formal setup. We defined an elementary statement to be of the from $a \in A$ with a an object and A a set. Now an eleemntary statement will be of the form $a \in b$ with a and b both sets. Rather than runniognt hrough the definitions again at this time, we simply declare that every set is an object and every object is a set. We will use lower case letters for sets. With this conbvention we can largely recycle the above discussion.

We can now give the equality symbol a precise meaning for all sets.

Definition 5 *The statetement $a = b$ is defined to be the statement*

$$\forall x : ((x \in A) \Leftrightarrow (x \in B))$$

Two sets arevequal when they contain the same elements. The naive intuition that there is only one set containing a given collection of elements. However, without a naive intuition about the equality sign there is no guarantee for thet. We could have two different sets a, b which contain the same elements, then the statement $a = b$ would be true but without any a priori prejudice about what thge sign $=$ should mean, we could not defer that they are the same set. It is difficult to resolve thsi dispute in the formal language entirely, but at least we should make sure for practiocal purposes a and b are the same, so we will declare

Axiom 1 (Extension) *The following statement is true.*

$$\forall x(\forall y(\forall z(((y = z) \wedge (y \in x)) \Rightarrow (z \in x)))))$$

Now allowing sets on both sides of $\in$ allows for some dangerous statements of the type

$$a \in a$$

that if true would together with some further simple concepts lead to paradoxa of Russell's type. For example the concept of "set" is a concept, and we might wnat to honor this by having a set a_0 which contains all sets. Hence

$$\forall a(a \in a_0)$$

is true. Now conceptually one may then wonder whether a set contains itself or not, so one may want to form a set a_1 such that

$$\forall a((a \in a_1) \Leftrightarrow (\neg(a \in a)))$$

is true. Now Russell's paradox is that inserting a_1 for a in the above, gives

$$(a_1 \in a_1) \Leftrightarrow (\neg(a \in a))$$

which is an absurd statement; exactly one of the two is supposed to be true.

To resolve this issue, we will not allow thinsg such as $a \in a$ to be true. Precisely, we adopt the axiom credited to von Neumann

Axiom 2 (von Neumann) *The following statement is true*

$$\forall x(\exists y((x = \emptyset) \vee (\forall z((y \in x) \wedge (\neg(z \in x) \wedge (z \in y))))))$$

This axiom says that every nonempty set contains an element which is disjoint from the original set. In particular for every given set a we can apply the axiom to the set $\{a\}$ (assuming that forming such set is legal, which will eb some other axiom), obtaining that a is disjoint from $\{a\}$ and thus a does not contain a.

Now clearly "set" is a concept and we may want to honmor by forming a set a_0 containing precisely all sets. However, this set would contain itself, contradicting the von Neumann axiom.

To salvage talking about the concept of sets, oine introduces some "mega sets", which are called classes. The purpose of classes is to describe concepts about sets. Hence there is a class of all sets. We will use upper case letters for classes. However, classes are a generalization of sets, so every set will also be a class.

Now there are classes that are sets and classes that are not sets. The latter are called proper classes. We say $a \in A$ is a statement provided a is a set and A is a class. Thus proper classes are not allowed to be on the left of the symbol $\in$.

If we are not allowed to put classes into classes, one may wonder how to express concepts about classes. To alleviate the concern, note that we have in some ways turned around the world. We started with objects and sets, where sets were allowed to be on both sides fo $\in$ while objects were only allowed on the left. Now we have sets and proper classes, and sets are allowed on both sides of the sign $\in$ while proper classes are only allowed on the right. This is a completely symmetric situation. Since we introduced sets as concepts about objects, one may dually also view sets as concepts about classes. So our language will be rich enough to formulate our desired concepts about classes.

With this preparation we are now ready to state the formal axioms of about sets and classes. We follow largely Gödel's 1940 paper.

There are classes A. A primitive statement is of the form $A \in B$ for two classes A, B. A primitive statement is true or false. A statement is any expression that can be formed recursively with the following steps

1. If P is a statement, then $\neg P$ is a statement. It is true if P is false and false if P is true
2. If P and Q are statements, then $(P) \wedge (Q)$ is a statement. It is true of both P and Q are true, and it is false otherwise.
3. If $P(x)$ is a statement involving a variable X for a class, then

$$\forall X(P(X))$$

is a statement. It is true if $P(X)$ is true for all classes.

We can also form derived statements using other symbols as discussed before

Definition 6 *The statement* Set(A) *is defined to be the statement*

$$\exists X(A \in X)$$

and if this statement is true A is called a set.

Definition 7 (Equality of classes) *The statetement $A = B$ is defined to be the statement*

$$\forall X : ((X \in A) \Leftrightarrow (X \in B))$$

Axiom 3 (Extension for classes) *The following statement is true.*

$$\forall X(\forall Y(\forall Z(((Y = Z) \wedge (Y \in X)) \Rightarrow (Z \in X)))))$$

We begin with producing sets. The first one is superfluous but convenenient to state at this time. It guarantees existence of the empty set

Axiom 4 (Emptyset) *The following is true*

$$\exists X(\forall Y((M(X)) \wedge (\neg(Y \in X))))$$

We'll pick once and for all an empty set and write $\emptyset$ for it. It is actually unique in the sense of the axiom of extension, as discussed before.

The following says that for any two sets there exists a set whose elements are precisely the two given sets.

Axiom 5 (Sets of two elements) *The following is true*

$$\forall X(\forall Y(\exists Z(\forall U((M(X) \wedge M(y)) \Rightarrow ((U \in Z) \Rightarrow (U = X) \vee (U = Y))))))$$

Definition 8 *if x and y are sets, thenw e write $\{x, y\}$ for the set guaranteed to exist by the last axiom. If $x = y$, then the set $\{x, x\}$ contains only one element (use extsnion) and we write $\{x\}$ for it.*

By this definiiton, we are as of yet allowed to use the set brackets as long as we write either one or two elements inside. The two elements ina set containing two elements are not ordered. To define an ordered pair (and triple) of elements, we define

Definition 9 *For any two sets x, y we write $\langle x, y\rangle$ for the set $\{x, \{x, y\}\}$ Given three sets x, y, z we write $\langle x, y, z\rangle$ for $\langle x, \langle y, z\rangle\rangle$.*

Note that an ordered pair is always a set containing two elements, even if it is the pair $\langle x, x\rangle$. By induction one can define tuples, i.e. the bracket $\langle x_1, x_2, \ldots, x_n\rangle$ for any number $n > 3$ of elements. To extend this definiiton to a bracket of one element one may define $\langle x\rangle = x$.

Next comes a list of axioms guaranteeing existence of classes. Some of them morrow logical connectives.

The following styatement says that there exists a class of ordered pairs (as a matter of fact it may contain more elements than ordered pairs, but only the ordered pairs matter) such that and ordered pair of sets $\langle x, y\rangle$ is in the set if and only if $x \leq y$. Thus the class describes the concept of ϵ within the realm of sets.

Axiom 6 (Epsilon class) *The following is true*

$$\exists X(\forall Y(\forall Z((M(Y) \wedge M(Z)) \Rightarrow (\langle Y, Z\rangle \in X) \Leftrightarrow (Y \in Z))))$$

The next axiom allows to take the intersection of two classes. It mirrors the logical concept of $\wedge$ on the statements concernign sets.

Axiom 7 (Intersection) *The following statement is true.*

$$\forall X(\forall Y(\exists Z(\forall U((U \in Z) \Leftrightarrow ((U \in A) \wedge (U \in Z))))))$$

The next axiom allows to take complements within the sets. In particular it imply that since the empty set is a class, there exists a class of all sets. It mirrors the concept of negation in the realm of statements about sets.

Axiom 8 (Complement) *The following statement is true.*

$$\forall X(\exists Y(\forall U((U \in Y) \Leftrightarrow (M(Y) \wedge (\neg(U \in A))))))$$

The following concerns classes of ordered pairs, which could be considered relations among sets. It says that given such a relation, there exists a set which contains all elements which ever occur as the second entry of one of the ordered pairs.

Axiom 9 (Domain) *The following statement is true.*

$$\forall X(\exists Y(\forall U((U \in Y) \Leftrightarrow (M(U) \cap (\exists Z)(M(Z) \cap \langle Z, U\rangle \in X)))))$$

The next axiom allos for any class A to take the cartesian product $B \times A$ (that is a set of ordered pairs) where B is the class of all sets. This direct product itself is a class of course. The direct product is not unique, as the axiom only concerns the ordered pairs within the direct product.

Axiom 10 (Direct product) *The following statement is true.*

$$\forall X(\exists Y(\forall Z(\forall U((M(Z) \wedge M(U)) \Rightarrow (((\langle Z, U\rangle \in Y) \Leftrightarrow (U \in A))))))$$

The next three axioms simply state that givena class of ordered pairs (triples), there is another class which contains exactly these pairs (triples) but in p[ermuted order. Since the group of permutations in three elements is spanned by two permutations, we have two axioms concerning triples. Again, all classes calimed to exist are not unique since the statements are only about the ordered pairs or triples within these classes.

Axiom 11 (Pair inversion) *The following statement is true.*

$$\forall X(\exists Y(\forall Z(\forall U((M(Z) \wedge M(U)) \Rightarrow (\langle Z, U\rangle \in Y \Leftrightarrow (\langle U, Z\rangle \in X))))))$$

Axiom 12 (Triple permutation I) *The following statement is true.*

$$\forall X(\exists Y(\forall Z(\forall U(\forall V(((M(Z) \wedge M(U)) \wedge M(V)) \Rightarrow (\langle Z, U, V\rangle \in Y \Leftrightarrow (\langle Z, V, U\rangle \in X))))))$$

Axiom 13 (Triple permutation II) *The following statement is true.*

$$\forall X(\exists Y(\forall Z(\forall U(\forall V(((M(Z) \wedge M(U)) \wedge M(V)) \Rightarrow (\langle Z, U, V\rangle \in Y \Leftrightarrow (\langle U, V, Z\rangle \in X))))))$$

The next group of axioms concerns the construction of relatively large sets.

The first one simply states that there is an infinite set, it resembles the corresponding Peano axiom for the natural numebrs though it does produce a unique set.

Axiom 14 (Infinite set) *The following is true.*

$$\exists X((M(X) \cap (X \neq \emptyset)) \cap (\forall Y((Y \in X) \Rightarrow (\exists Z((Y \in Z) \wedge (Z \in X))))))$$

The next set allwos to take arbitrary unions over sets, that is givena set of sets,one can take the union of all sets in the set.

Axiom 15 (Union over a set) *The following is true.*

$$\forall X(\exists Y(M(Y) \cap \forall Z(\forall U(((Y \in Z) \wedge (Z \in X)) \Rightarrow (X \in Z)))))$$

The next axiom asserts that for any set there is a set which contains all subsets of the first set. The way the axiom is formulated the power set may contain

Axiom 16 (Power set) *The following is true.*

$$\forall X(\exists Y(M(Y) \cap \forall Z((\forall U((U \in Z) \Rightarrow (U \in X))) \Rightarrow (Z \in Y))))$$

Remark: It seems that most of ordinary mathematics can essentially still be doen if one drops the demand $M(Y)$ thus only obtaining a power class. This would then be cosnisting with demanding that all sets are countable.

The next axiom is one of substituting some variable by means of a function. We first need to describe what it means to be a function. Informally it says a function is a set of ordered pairs such that

Definition 10 *The statement $F(A)$ is defined to mean*

$$\forall X(\forall Y(\forall Z(((M(X) \cap M(Y)) \wedge M(Z)) \Rightarrow (((\langle Y, X\rangle \in A) \cap (\langle Z, X\rangle \in A)) \Rightarrow (X = Y))))$$

Axiom 17 (Substitution) *The following is true.*

$$\forall X(M(X) \Rightarrow (\forall Y(F(Y) \Rightarrow (\exists Z(M(Z) \cap \forall U((U \in Z) \Leftrightarrow (\exists V((V \in X) \wedge \langle U, V\rangle \in Y))))))))$$

Finally we reiterate the axiom of von Neumannn for classes

Axiom 18 (von Neumann) *The following statement is true*

$$\forall X(\exists Y((X = \emptyset) \vee (\forall z((y \in x) \wedge (\neg(z \in x) \wedge (z \in y))))))$$

and we state the axiom of choice....

Axiom 19 *The following is true*

$$\exists X(F(X) \wedge \forall Y(M(Y) \Rightarrow ((Y = \emptyset) \rightarrow (\exists Z((Z \in Y) \wedge (\langle Z, Y\rangle \in X))))))$$

This says thre is a class which for every set chooses an element in the set.

Cardinality of sets

Recall that a function f from a set X to a set Y is a relation xfy with the properties

1. For all $x \in X$ there exists $y \in Y$ such that xfy
2. If for $x, x' \in X$ and $y \in Y$ with xfy and $x'fy$ we have $x = x'$.

   we have xfy and $x'fy$ then $x = x'$

Informally, for every $x \in X$ thyere exist (first property) a unique (second property) $y \in Y$ such that xfy

A function is called

1. surjective, if for all $y \in Y$ there exists $x \in X$ such that xfy
2. injective, if for every $y, y' \in Y$ and $x \in X$ with xfy and xfy' we have $y = y'$.

Note the symemtry betwen the definition of a function and these two properties. In particualr, if and only if a function f is injective and surjective, then inverse relation g defined by ygx if and only if xgy is a function. In thsi case, the inverse relation is injective and surjective as well.

A function is called bijective, if it is injective and surjective.

We writing $y = f(x)$ for the relation xfy.

Let A_0 denote the empty set and assume A_n is already defined then define $A_{n+1} = A \cup \{n\}$. Informally, at elats for $n > 0$, $A_n = 0, 1, 2, \ldots, n-1$ and there are n elements in A_n.

Lemma 1 *Let $n, m \in \mathbf{N}$.*

1. *There is an injective function from A_n to A_m if and only if $n \leq m$*
2. *Let $n \neq 0$. There is a surjective function from A_m to A_n if and only if $n \leq m$.*

Note that there is a surjective map from A_m to A_0 if and only if $m = 0$. hecne the caes $n = 0$ is a true exception to the scheme, but since this exception only concerns the empty set we shall not be too troubled by that.

Proof: Assume $n \leq m$. then an injective map from $A_n \to A_m$ is given by $f(k) = k$ for every $k \in A_n$. If $n \neq 0$ and thus $1 \in A_n$, then a surjective map from A_m to A_n is given by $f(k) = k$ if $k \in A_+m \cap A_n$ and $f(k) = 1$ if $k \in A_m \setminus A_n$.

Now let $n > m$. We prove non-existence of an injective map $A_n \to A_m$ by induction on n. If $n = 0$, then there is nothing to show. If $n = 1$, then $m = 0$. Since A_m is empty, there is no pair $(0, k)$ with $k \in A_m$ and hecne we cannpot satisfy the first property of a function for the element $0 \in A_n$. Assume we have proven the statement for n. Assuem to get a contradiction that we have an injective map form $A_n to A_m$. ...

□

Exercise 2 *1. An injective map from A_n to A_n is surjective.*

2. A surjective map from A_n to A_n is injective.

Definition 11 *A set S is called finite, if there is some $n \in \mathbf{N}$ and a bijection from S to A_n. If such bjection exists, S is said to have the same cardinality as A_n*

Exercise 3 *Given a finite set, there is exactly one $n \in \mathbf{N}$ such that S has the same cardinality as A_n.*

One may also say simply that S has cardinality n.

The set $\mathbf{N}$ is not finite. Namely, if it was, then there was an injective map from $\mathbf{N} \to A_n$ for some n. This map could be restricted to A_{n+1}, giving an injective map from A_{n+1} to A_n, which is impossible.

We have the following observations, destroying hope to have staright forward generalizations of the theorems for finite sets.

Lemma 2 *1. There is an injective map from $\mathbf{N}$ to $\mathbf{N}$ that is not surjective.*

2. There is a surjective map from $\mathbf{N}$ to $\mathbf{N}$ which is not injective.

Proof: The map ν is injective (one Peano axiom) but not surjective (0 is not in the range, as another Peano axiom states.) Defien a map f as follows: $f(0) = 0$ and whenever n is of the form $\nu(m)$ (that is whenevr $n \neq 0$), then m is unique and we define $f(n) = m$. then 4f4 is surjective but not injective since 0 has two preimages. $\square$

Definition 12 *Two sets are said to be equivalent, if there exists a bijection between the two sets.*

The follwing will show that not all infinte sets are equivalent.

Lemma 3 *For every set S, there is no bijection from S to the set of all subsets of S*

Proof: This is called the Cantor diagonal element. Assume to get a contradiction that there is such a bijection f. Let T be the set of all elements $s \in S$ in T which satisfy $s \notin f(s)$. By surjectivity, there is a $t \in S$ such that $f(t) = T$. If $t \in T$, then $t \in f(t)$, a contradiction. If $t \notin T$, then $t \notin f(t)$ and hence $t \in T$, a contradiction. $\square$

We define a relation of sets as follows: $S \leq T$ if there exists an injective map $f : S \to T$. Note that the collection of all sets is not a set but what is called a class. This is therefore not an order relation on a set, but on a class.

Lemma 4 (Schröder Bernstein) *This relation is a partial order in that it satisfies*

1. *Reflexivity: $S \leq S$ for all sets S.*
2. *Antisymmetry: $S \leq T$ and $T \leq S$ implies $S = T$.*
3. *Transitivity $S \leq T$ and $T \leq U$ implies $S \leq U$.*

Proof: next time