0.1 Compactness

Definition 1 A metric space is called compact, if every cover of the space has a finite subcover.

As in the case of separability we have the following two observations:

Lemma 1 Finite sets are compact.

Lemma 2 If a metric subspace $M'$ of a space $M$ is compact and we are given any collection of balls in $M$ that cover $M'$, then there is a finite subcover.

The analogy of the theories of separable and compact sets has an end when the countability of the set of radii $D$ comes into play. For example, compact spaces do not have finite dense sets (finite sets are never dense in any larger space). Also, a subset of a compact set need not be compact.

On the other hand one has some immediate consequences from applying the compactness property to all balls centered at a given point:

Definition 2 A nonempty metric space $(M, d)$ is bounded if it is empty or there exists a ball which contains the whole space $M$.

This is clear if $M$ is empty by using the empty ball. Note that if $K$ is a nonempty subset of another metric space $M$, then it suffices to find a ball in $M$ that covers $K$. Namely, if $B_r(y)$ covers $K$ and $x \in K$, then $B_{2r}(x)$ also covers $K$ by the triangle inequality.

Lemma 3 A compact metric space $(K, d)$ is bounded.

Proof: We may assume $K$ is not empty. Pick a point $x \in K$ and observe that

$$M = \bigcup_{r \in D} B_r(x)$$

, since every point $y$ in $K$ has some finite distance $d(x, y)$ to $x$ and there exists $r \in D$ with $d(x, y) < r$ and hence $y \in B_r(x)$. Hence the collection of balls about $x$ covers
the compact set, and hence there is a finite collection of balls about \( x \) which cover the compact set. Since the balls are nested, it suffices to take the largest ball to cover the compact set. This proves boundedness. \( \square \)

There is a dual observation using complements of balls

**Lemma 4** Every compact metric space \((K, d)\) is complete.

Note that the notion of completeness is also independent of the embedding.

**Proof:**
Pick a cone \( B \) in \( K \). For every \((x, r) \notin B\) there exists \( \epsilon > 0\) and \((y, s) \in B\) such that

\[
d(x, y) \geq r + s + \epsilon
\]

Pick the ball \( B_r(r) \) and let \( C \) be the collection of all these balls.

We have two cases:

1. Assume \( C \) does cover \( K \). Then there is some finite subcover. Each ball in the cover comes with a point \((y, s) \in B\) as above, pick \( s_0 \) the minimal \( s \). If \( s = s_0 \), we are done since then the cone has a tip. We claim \( s_0 > 0 \) is impossible. For assume to get a contradiction that \( s_0 > 0 \). Pick \((z, t) \in B\) with \( t < s_0 \). The point \( z \) is in the finite subcover. Let \( B \) be the ball in that cover containing \( z \), it comes with a point \((x, y) \notin B\) and \((y, s) \in B\) with

\[
r + s + \epsilon \leq d(x, y)
\]

Then

\[
r + t + \epsilon \geq d(x, z) + d(y, z) \geq d(x, y) \geq r + s + \epsilon
\]

This is a contradiction since \( t < s_0 \leq s \).

2. Assume \( C \) does not cover \( K \), and let \( z \) be a point not covered. Then there does not exist \((y, s) \in B\) with

\[
d(z, y) \geq r
\]

Hence the cone is of the form \( B(z) \)

The following definition is a property between separability and compactness.

**Definition 3** A subset \( M' \) of a metric space \( M \) is called totally bounded, if for every \( r > 0 \) there exists a cover of \( M \) by balls of radius \( r \).

**Lemma 5** A totally bounded space is separable.

**Proof:** For each \( r \) pick a finite cover \( C_r \) of the space \( M \). Let \( C \) be any cover of \( M \). Let \( x \) be a point in \( M \) and \( B_r(y) \) a ball in \( C \) containing \( x \). then there is a ball \( B_s(x) \) contained in \( B_r(y) \). Pick a ball \( B^x \) in \( C_{s/2} \) containing \( x \), then this ball is contained in \( B_r(y) \). The collection of all \( B^x \) is countable since contained in the countable union of finite sets \( C_r \). Picking for each ball \( B^x \) a ball in \( C \) that contains \( B^x \) gives a countable subcover of \( C \). \( \square \)
Lemma 6 A compact metric space is totally bounded.

Proof: Apply the definition of compactness to the collection of balls of radius $r$. □

Lemma 7 Let $(K,d)$ be a totally bounded complete metric space. Then $K$ is compact.

Proof:
Denote by $C_n$ a finite cover of $M$ by balls of radius $2^{-n}$. Pick a ball in $C_{2^{-n}}$ that is not covered by finitely many balls in $C$. Assume we have picked a ball in $C_{2^{-n}}$ not covered by finitely many balls in $C$, then cover this balls by finitely many balls in $C_{2^{-n+1}}$ and pick one such ball which is not covered by finitely many balls in $C$.
This gives a sequence of balls. Let $x_n$ be the sequence of centers and note that $(x_n, 2^{-n})$ is a pre-cone. Let $x$ be the tip of the corresponding cone.
Let $B_r(y)$ be a ball in $C$ that contains $x$. Then this ball contains a ball $B_s(x)$. Then this ball contains the selected ball $C_{2^{-n}}$ for some $n$ with $2^{-n} < s/2$. This is a contradiction to the choice of these selected balls.

□

Totally boundedness relates to Cauchy sequences as follows:

Lemma 8 Let $M$ be a totally bounded metric space. Then every sequence has a subsequence which is Cauchy.

Proof: Let $f : N \to M$ be a sequence. Define $f_0 = f$. Assume we have defined $f_n$ for $n \geq 0$. Define a subsequence $f_{n+1}$ of $f_n$ so that $f_{n+1}(m)$ coincides with $f_{n+1}(m)$ for $m \leq n$, and so that $f_{n+1}(m) \in B_{2^{-n}}(x)$ for some ball of radius $2^{-n}$. This is possible, since we can cover the space $M$ by finitely many balls, so at least one of them needs to contain infinitely many values of the sequence $f_n$. Then define $g(n) = f_n(n)$. This is a subsequence of $f$, and it is Cauchy. □

Lemma 9 Let $M$ be a metric space. Assume that every sequence $f : N \to M$ has a subsequence that is Cauchy, then $M$ is totally bounded.

Proof: Fix $r$ and pick a point $x_0 \in M$. Assume we have already picked $x_n$. If the collection of balls $B_r(x_k)$ with $0 \leq k \leq n$ covers $M$, we are done having constructed a finite cover of $M$. If not, pick $x_{n+1}$ not covered by these balls. We claim this process cannot go on indefinitely. If it does, then we have a sequence $x_n$ which has no Cauchy subsequence since any two elements in the sequence have distance at least $r$. This is a contradiction. □

Lemma 10 Let $(M,d)$ be a compact metric space and $M'$ be a subspace. If $M/M'$ is the union of balls, then $M'$ is compact.

Proof: Let $C$ be a cover of $M'$. Since $M \setminus M'$ is the union of balls, the set $C'$ of these balls forms a cover of $M \setminus M'$. Hence $C \cup C'$ covers $M$. Find a finite subcover. Remove all balls in $C'$ from this finite subcover, then it still covers $M'$. □
Exercise 1 Let \((M,d)\) be a complete metric space and \(M'\) be a subspace. If \(M/M'\) is the union of balls, then \(M'\) is complete.

Note that \(\mathbb{R}^n\) with Euclidean metric is complete.

Lemma 11 Bounded complete subsets of \(\mathbb{R}^n\) with Euclidean metric are compact.

Proof: We prove that every sequence in \(K\) has a subsequence which is Cauchy. Let \((x_1, \ldots, x_n)\) denote the sequence. Since the sequence is bounded, each entry is bounded. Choose a subsequence so that \(x_1\) is monotone. Then choose a subsequence so that \(x_2\) is monotone, and so on. Hence we may assume all sequences are monotone. But then all coordinate sequences are Cauchy, and hence the full sequence is Cauchy.

Since \(K\) is complete, it has a limit.

Exercise 2 The space \(l^2(\mathbb{N})\) is separable and complete.

Lemma 12 The ball \(B_1(0)\) in \(l^2(\mathbb{N})\) is not compact.

Proof: Let \(f_n\) be the element in \(l^2(\mathbb{N})\) which satisfies \(f_n(m) = 0\) if \(n \neq m\) and \(f_n(m) = 1\) if \(n = m\). Then Any two elements have distance \(> 1\), hence this sequence has no subsequence that is Cauchy.

Exercise 3 Let \(a : \mathbb{N} \to \mathbb{R}^+\) be a sequence with \(\limsup_{n \to \infty} a_n = 0\). Let \(K\) be the set of all sequences \(f\) in \(l^2(\mathbb{N})\) (with values in \(\mathbb{R}\)) such that

\[
\sum_{n=0}^{\infty} a(n)f(n)^2 < \infty < \infty
\]

Prove that \(K\) is compact.

Exercise 4 Prove the converse of the above. If \(K \subset l^2(\mathbb{N})\) is compact, then there exists a sequence \(a_n\) as above such that \(\sum_{n=0}^{\infty} a(n)f(n)^2 < \infty\) for all \(f \in K\).

The following is a version of the fact that finite subsets in a totally ordered space have a maximum and a minimum.

Lemma 13 Let \((M,d)\) be a metric space and let \(K\) be a subspace. Let \(x \in M\), then there exists \(y\) and \(z\) in \(K\) such that

\[
d(x,y) = \inf_{t \in K} d(x,t)
\]

\[
d(x,y) = \sup_{t \in K} d(x,t)
\]
Proof: There is a sequence \( t_n \) of elements in \( K \) such that \( d(x, t_n) \) is monotone increasing and
\[
\sup_n d(x, t_n) = \sup_{t \in K} d(x, t) = S
\]
Pick a subsequence that is Cauchy. Then this sequence has a limit \( t \in K \). Then we have
\[
d(x, t) \geq d(x, t_n) - d(t, t_n) \geq S - \epsilon - \epsilon
\]
for arbitrarily small \( \epsilon \). Hence
\[
d(x, t) = S
\]
The proof of existence of a minimum is similar. \( \square \)

**Definition 4** Let \((M, d)\) and \((M', d')\) be two metric spaces. The Cartesian product of the two metric spaces is the set of all ordered pairs \((x, x')\) with \(x \in M\) and \(x' \in M'\) and distance
\[
d((x, x'), (y, y')) = \max \{d(x, y), d(x, y')\}
\]

**Exercise 5** Prove that this is indeed a metric space.

**Exercise 6** The Cartesian product of two separable metric spaces is separable.

**Lemma 14** The Cartesian product of two compact metric spaces is compact.

Proof: It suffices to find finite subcovers of covers by balls in this metric. Suppose we are given such a cover. For each \(x \in M\) this covers in particular the set \(\{x\} \times M'\). Find a finite subcover. Pick the minimal radius \(r\) of this finite subcover. Then this finite subcover covers \(B_r(x) \times M'\). Now these \(B_r(x)\) cover all of \(M\). Find a finite subcover. This produces a finite collection of finite sets covering all of \(M \times M'\) \( \square \)

**Lemma 15** Assume \((K, d)\) is a compact metric space and \((M, d')\) is a metric space. Assume \(f: K \to M\) is continuous. Then the range of \(f\) is compact.

Proof: Let \(C\) be an open cover of \(f(K)\). We produce a cover of \(K\). Let \(x \in K\). Let \(B_x\) be a ball in \(C\) that contains \(f(x)\). By continuity of \(f\), there is a ball \(B_x\) in \(K\) such that \(f(B_x) \subset B_x\). The collection of these balls \(B_x\) covers \(K\). Pick a finite subcover. Consider all \(B_x\) corresponding to the chosen \(B_x\). This is a finite collection. We claim it covers \(f(K)\). Let \(y \in f(K)\). The there is \(z\) such that \(f(z) = y\). Let \(B_z\) be the chosen ball containing \(z\). Then \(B_z\) contains \(y\). \( \square \)

**Lemma 16** Let \((K, d)\) be a compact space and \((M, d')\) be a metric space. Let \(f: K \to M\) be continuous. Then \(f\) is uniformly continuous.

Fix \(s > 0\). define for each \(x \in K\) \(r(x)\) to be the supremum of all numbers \(r \leq 1\) such \(d(x, y) \leq r\) implies \(d'(f(x), f(y)) \leq \delta\). We claim that \(r\) is continuous, and first see how this will finish the proof. If continuous, then it attains its minimum on the compact set \(K\). The minimum cannot be 0 by continuity at that point. Hence the minimum is larger than zero, which proves uniform continuity since \(s\) was arbitrary.

Now we turn to the proof of continuity. Fix \(x\) and note that in the vicinity of \(x\) we will use.
Arzela Ascoli

Lemma 17 Let $M$ and $M'$ be compact spaces. Assume $F \subseteq C(M, M')$. Assume that there is a modulus of continuity $g$ such that all $f \in F$ satisfy

$$g(d'(f(x), f(y))) \leq d(x, y)$$

Then $F$ is totally bounded.

Proof:

Pick $r > 0$. Pick $s = g(r/4)$. Cover $K$ by finitely many balls $B_{s}(x_{i}), i = 1, \ldots, n$. For each $i$ cover the compact set $\{f(x_{i}), f \in F\}$ by finitely many balls $B_{r/4}(y_{i,j})$, $j = 1, \ldots m$.

For each of the finitely many functions $h : \{1, \ldots, n\} \rightarrow \{1, \ldots, m\}$ for which there exists a function $f \in F$ with

$$f(x_{i}) \in B_{r/4}(y_{i,h(i)})$$

Pick one of these functions and call it $f_{h}$. This is a finite collection in $F$. We claim the corresponding balls of radius $r$ about these functions in $C(M, M')$ cover $F$.

To see this, let $f$ be any function in $F$. Pick a function $h$ such that

$$f(x_{i}) \in B_{r/4}(y_{i,h(i)})$$

which exists by construction. We aim to show for all $x \in M$ that

$$d(f(x), f_{h}(x)) \leq r$$

Given such $x$, there is $x_{i}$ with $x \in B_{s}(x_{i})$. Then

$$d(f(x), f'(x)) \leq d(f(x), f(x_{i})) + d(f(x_{i}), y_{i,h(i)}) + d(y_{i,h(i)}, f_{h}(x_{i})) + d(f_{h}(x_{i}), f_{h}(x))$$

But each of these numbers are less than $r/4$ by construction. This completes the proof.

The Riemann integral for continuous functions

So far we have defined the integral

$$\int_{0}^{x} f(t) \, dt$$

for monotone increasing functions. It satisfies

$$\int_{0}^{x} (f(t) + g(t)) \, dt = \int_{0}^{x} f(t) \, dt + \int_{0}^{x} g(t) \, dt$$
\[ \int_0^x f(t) \, dt \leq \int_0^x g(t) \, dt \]

whenever \( f \leq g \).

We can extend this to the space of functions of bounded variation. This is the set of pairs \((f, g)\) of monotone increasing functions with equivalence relation \((f, g) \sim (f', g')\) if \( f + g' = f' + g \).

Recall that these functions take value in the real numbers, which is the set of pairs \((x, y)\) with the equivalence relation \((x, y) \sim (x', y')\) if \( x + y' = x' + y \).

The Riemann integral is then defined as

\[ \int_0^x (f(t), g(t)) \, dt = (\int_0^x f(t) \, dt, \int_0^x g(t) \, dt) \]

It does not depend on the choice of representative of the equivalence class, since for any two representatives \((f, g), (f', g')\) with \( f + g' = f' + g \) we have

\[ \int_0^x f(t) \, dt + \int_0^x g'(t) \, dt = \int_0^x f(t) + g'(t) \, dt \]

\[ \int_0^x f'(t) + g(t) \, dt = \int_0^x f'(t) \, dt + \int_0^x g(t) \, dt \]

and hence

\[ (\int_0^x f(t) \, dt, \int_0^x g(t) \, dt) \sim (\int_0^x f'(t) \, dt, \int_0^x g'(t) \, dt) \]

We refer to introduction of real numbers to elsewhere in the script and use their properties.

We like to extend this integral on \( BV([0, x]) \) to the space \( C([0, x], \mathbb{R}) \). More precisely, neither of the spaces contains the other, even after natural identification of \((f, g) \in BV([0, x])\) with the function \( f - g \in C([0, x])\). However, the space of Lipschitz functions is in the intersection of both. Clearly Lipschitz functions are continuous. Moreover, if \( f \) is Lipschitz with constant \( L \), then we can write it as difference of two monotone functions as follows:

\[ f(t) = (f(t) + Lt) - Lt \]

The Lipschitz condition implies the first function is monotone.

Recall that the Lipschitz functions are dense in \( C([0, x]) \).

The point is that for Lipschitz functions we have the continuity property (actual even a Lipschitz continuity)

\[ |\int_0^x f(t) \, dt - \int_0^x g(t) \, dt| \leq x \|f - g\|_\infty \]

This can be seen as follows. Let \( L \) be the maximum of the Lipschitz constants for \( f \) and \( g \). Then we write

\[ f(t) = (f(t) + L(t)) - L(t) \]

\[ g(t) = (g(t) + L(t)) - L(t) \]
Representing each as monotone function. Then the LHS can be seen to be
\[
\left| \int_0^x f(t) + Lt \, dt - \int_0^x g(t) + Lt \, dt \right|
\]
Denote \( \| f - g \|_\infty \) by \( d \), then
\[
\int_0^x f(t) + Lt \, dt \leq \int_0^x g(t) + Lt + d \, dt = \int_0^x g(t) + Lt \, dt + xd
\]
\[
\int_0^x g(t) + Lt \, dt \leq \int_0^x f(t) + Lt + d \, dt = \int_0^x f(t) + Lt \, dt + xd
\]
This proves the desired Lipschitz continuity.

Since Lipschitz functions are dense in \( C([0, x]) \), there is a unique extension to \( C([0, x]) \) if this Lipschitz mapping. This defines
\[
\int_0^x f(t) \, dt
\]
for functions \( f \) in \( C([0, 1]) \)

Similarly one can see that the \( n \)-th Lower and upper Riemann sums can be extended to \( C([0, 1]) \). Moreover, all the \( n \)-th Riemann sums for a fixed function are bounded by \( \| f \|_\infty x \) and Lipschitz with constant \( x \). This makes them an equicontinuous family.

Since for every Lipschitz function \( f \) the Riemann sums converge to the integral, so do the Riemann sums for arbitrary functions in \( C([0, 1]) \) (use a theorem about uniqueness of the limit.)

Let \( I : C([0, 1]) \rightarrow C([0, 1]) \) be defined by
\[
I f(x) = \int_0^x f(t) \, dt
\]
First of all, we note that indeed \( I f \) defines a continuous function, indeed, it is even Lipschitz. But if \( x < y \) we have
\[
I f(y) - I f(x) = \int_x^y f(t) \, dt \leq |y - x| \| f \|_\infty
\]
Hence \( I f \) is even a Lipschitz function.

We like to understand properties of the range of the map \( I \).

**Definition 5** We call a function \( g : [0, 1] \rightarrow \mathbb{R} \) to be continuously differentiable, if for each \( x \in [0, 1) \) the limit
\[
g(x) = \lim_{n \rightarrow \infty} \frac{g(x + 2^{-n}) - g(x)}{2^{-n}}
\]
exists and \( g \) is a uniformly continuous function on \([0, 1]\).
Lemma 18  The range of \( I \) is contained in the space of continuously differentiable functions. I.e., If \( f \) is continuous and \( g = If \), then for each \( x \in [0, 1) \) we have

\[
\lim_{n \to \infty} \frac{If(x + 2^{-n}) - If(x)}{2^{-n}} = f(x)
\]

Proof: Note if \( n \) is large enough then

\[
2^{-n}(f(x) - \epsilon) \leq If(x + 2^{-n}) - If(x) \leq 2^{-n}(f(x) + \epsilon)
\]

This proves existence of the limit. Since \( f \) is uniformly continuous, this proves the lemma. \( \Box \)

Lemma 19  The map \( I \) is injective.

Proof: Suppose \( If = Ig \). Then \( I(f - g) = 0 \). Hence it suffices to show \( Ih = 0 \) implies \( h = 0 \). Assume to get a contradiction \( h \neq 0 \), then there exists \( y \in (0, 1) \) with \( h(y) \neq 0 \). Assume wlog \( h(y) > \epsilon > 0 \). Then there is a ball of radius \( \delta \) about \( y \) with \( h(x) > \epsilon \) in this ball. But then

\[
Ifh(y - \delta) + 2\epsilon\delta \leq Ih(y + \delta)
\]

And thus \( IH \neq 0 \). This is a contradiction. \( \Box \)

Lemma 20  The map \( I \) is surjective.

Proof: Let \( g \) be some continuously differentiable function, and let \( f \) denote its continuous derivative (extended to \([0, 1]\)). It suffices to show \( g = If \), i.e., for every

\[
\int_0^x f(t) \, dt = g(x)
\]

\( \Box \)