Modeling the impact of birth control policies on China’s population and age: effects of delayed births and minimum birth age constraints

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Abstract

We consider age-structured models with an imposed refractory period between births. These models can be used to formulate alternative population control strategies to China’s one-child policy. By allowing any number of births, but with an imposed delay between births, we show how the total population can be decreased and how a relatively older age distribution can be generated. This delay represents a more “continuous” form of population management for which the strict one-child policy is a limiting case. Such a policy approach could be more easily accepted by society. Our analyses provide an initial framework for studying demographics and how social constraints influence population structure.

Keywords: population biology, demographics, McKendrick equation, population control, one-child policy

1 Introduction

Models of age-structured population dynamics are often based on the classic McKendrick equation [McKendrick, 1926, Kermack and McKendrick, 1927] (sometimes called von Foerster equation [von Foerster, 1959]). These equations describe the dynamics of the mean population as a function of time \( t \) and expressed as a density in age \( a \). The solutions to the McKendrick equations can be partially solved using the method of characteristics and numerical approximations [Perthame, 2007, Keyfitz and Keyfitz, 1997] across many contexts. Moreover, stochastic extensions to incorporate the random times of birth and death (demographic stochasticity) have been formulated using branching processes [Jiang et al., 2017] and kinetic and operator theory [Greenman and Chou, 2016, Chou and Greenman, 2016, Greenman, 2017, Xia and Chou, 2021]

Age-structured equations have been used to predict the evolution of human and animal populations [Tuljapurkar, 1983, Bongaarts and Greenhalgh, 1985, Feeney and Yu, 1987]. Using such models and ideas from control theory to frame population control strategies was vogue in the 1970s [Langhaar, 1972, Pollard, 1973, Falkenburg, 1973, Hritonenko and Yatsenko, 2010]. A profound example was its use in 1979 by Jian Song [Song, 1980, 1982], a Chinese engineer who numerically solved the one-component McKendrick equation using birth rates associated with China in the late 1970s (see Fig. 1). By projecting future populations associated with different birth rates (expressed by the mean number of children per woman), he found that in order to keep the population manageable (\(~700\text{ million} - 1\text{ billion}\)) within 100 years, this control parameter would have to be decreased to the point where each woman is allowed only one child [Song, 1980, 1982, Song et al., 1988]. This research provided the technical basis of the one-child policy in China [Greenhalgh, 2004, Bacaër, 2011].

In the 1970s, China had encouraged (but not enforced) people to marry later, wait longer before childbearing, and have fewer children (“later-longer-fewer” policy) [Bongaarts and Greenhalgh, 1985]. Despite concerns from social
scientists and demographers who proposed such “softer” controls, the one-child policy was implemented in 1980, based on the implications of Jian Song’s numerical solutions to the McKendrick equation. Rather than imposing a maximal number of children, a minimum delay between two consecutive births [Greenhalgh, 2004] or a minimum birth age could have been imposed. Such a policy would arguably have been more easily enforced and would have led to fewer unintended consequences such as a skewed sex ratio and an elder-heavy age distribution. Here, we retrospectively model such alternatives and make predictions as policies change.

Specifically, we extend the McKendrick age-structured model to incorporate a delay between successive births by each female. In order to do so, we must explicitly delineate individuals who have not given birth from those who have given birth at least once. Imposed delays between successive births can then be formally described by adjusting the birth rate function of the individuals who have given birth at least once. We solve our model equations using parameters appropriate to 1981 China and compare predictions of the graded policies with those of a strict one-child policy. We explore how the total population and age distribution are affected by different values of imposed refractory periods and minimum birth age.

2 Mathematical Model

When applying age-structured partial differential equation (PDE) models to two-sex populations, a simple assumption is to consider only the density of females at time $t$ with age $a$. The predicted number of females with age between $a$ and $a + da$ is thus $f(a, t)da$. Indeed, unless the female population is much larger than the male population (e.g., after a war), the female population can be considered as the “limiting quantity” that determines the number of births. In other words, the frequency of births in the total population is relatively insensitive to the male population. The McKendrick equations describing the female population density $f(a, t)$ are formulated as

$$\frac{\partial}{\partial t} f(t, a) + \frac{\partial}{\partial a} f(t, a) = -\mu_t(a) f(t, a)$$

(1a)

$$f(t, 0) = \eta \int_0^\infty \beta eff(a) f(t, a) da$$

(1b)

$$f(0, a) = I_t(a),$$

(1c)

where $\mu_t(a)$ represents the death rate of females of age $a$, $\beta eff(a)$ is the observed birth rate of women of age $a$, $\eta$ is the fraction of births that produce girls, and $I_t(a)$ is the age distribution of the initial population at $t = 0$.

Equation (1a) describes the time-evolution of the population, Eq. (1b) denotes the boundary condition at age $a = 0$ describing the number of girls born at time $t$, and Eq. (1c) specifies the initial condition. This model neglects the explicit mating-age male population which is valid when $\eta$ is maintained below 0.5, giving rise to more males than females. For humans, $\eta \approx 0.48 – 0.49$ naturally (but this is compensated by a slightly higher mortality in males across all ages). With sex-selective abortion, $\eta$ can be even smaller [Li and Meng, 2014]. If one were also interested in the male population $m(t, a)$, it would obey the same equations except with a male version of the death rate $\mu_m(a)$, an initial condition $I_m(a)$, and a boundary condition for male newborns: $m(t, 0) = (1 - \eta) \int_0^\infty \beta eff(a) f(t, a) da$.

2.1 Delayed birth model

Now, in order to introduce a delay between consecutive births, we need to further partition the female population into those who have never had a child and those who have already had a child (and who may need to wait a certain time before having another one). The population densities for each of these classes of females are defined as:

$f_0(t, a)$: the population density of childless females. The quantity $f_0(t, a)da$ is the number of females with age between $a$ and $a + da$ and who have never had a child up to the current time $t$.

$f(t, a, \tau)$: the population of females who have had at least one child. The quantity $f(t, a, \tau)da d\tau$ is the number of females at time $t$ whose age is between $a$ and $a + da$ and whose youngest child’s age is between $\tau$ and $\tau + d\tau$.

We will assume that these two populations have the same age-dependent death rate $\mu_t(a)$ but give birth at different rates $\beta_0(a)$ and $\beta(a, \tau)$, respectively. We also define the total female population density as
\[ f_{\text{tot}}(t, a) = f_0(t, a) + \int_0^a f(t, a, \tau) d\tau \] (2)\\

and the total number of females at time \( t \) as

\[ n(t) = \int_0^\infty f_{\text{tot}}(t, a) da = \int_0^\infty f_0(t, a) da + \int_0^\infty da \int_0^a d\tau f(t, a, \tau). \] (3)

The age-structured McKendrick equations for \( f_0 \) and \( f \) are:

\[ \frac{\partial}{\partial t} f_0(t, a) + \frac{\partial}{\partial a} f_0(t, a) = - (\mu(t) + \beta(t)) f_0(t, a) \] (4a)\\
\[ \frac{\partial}{\partial t} f(t, a, \tau) + \frac{\partial}{\partial a} f(t, a, \tau) + \frac{\partial}{\partial \tau} f(t, a, \tau) = - (\mu(t) + \beta(t, \tau)) f(t, a, \tau) \] (4b)\\
\[ f_0(t, 0) = \eta \left( \int_0^\infty \beta_0(a) f_0(t, a) da + \int_0^\infty da \int_0^a d\tau \beta(a, \tau) f(t, a, \tau) \right) \] (4c)\\
\[ f(t, a, 0) = \beta_0(a) f_0(t, a) + \int_0^a \beta(a, \tau) f(t, a, \tau) d\tau \] (4d)\\
\[ f_0(0, a) = I_0(a) \quad \text{and} \quad f(0, a, \tau) = I(a, \tau). \] (4e)

Equation (4a) describes the evolution of \( f_0 \) as in the classical McKendrick equation (cf Eq. (1a)) with birth rate \( \beta_0(a) \). For \( f(t, a, \tau) \), we must introduce the new variable \( \tau \) to mark the time since the last birth. This brings in another convection term in Eq. (4b) since \( \tau \) increases alongside time \( t \) and age \( a \). The birth rate \( \beta \) for this population can depend on both the age \( a \) and the time \( \tau \) since the last birth. Eq. (4c) gives the number of girls \( f_0(t, 0) \) born at time \( t \), while Eq. (4d) describes \( f(t, a, 0) \), the density of females at age \( a \) at time \( t \) who just gave birth. These individuals can arise from the \( f_0 \) population (females who have never had a child) or from the \( f \) population itself (females who have already had at least one child). Thus, the boundary conditions (4c) and (4d) couple the two populations \( f_0 \) and \( f \). Finally, equations (4e) simply describe the initial conditions for \( f_0 \) and \( f \).

In Appendix A, we explicitly show that the total female population density \( f_{\text{tot}}(t, a) \) (Eq. (2)) satisfies the standard age-structured McKendrick equation

\[ \frac{\partial}{\partial t} f_{\text{tot}}(t, a) + \frac{\partial}{\partial a} f_{\text{tot}}(t, a) = - \mu(t) f_{\text{tot}}(t, a). \] (5)

Within a model that explicitly considers the time \( \tau \) since the last childbirth, we can easily describe an imposed hypothetical policy that applies a refractory period \( \delta \) between births. After having a child and before this refractory period \( 0 \leq \tau \leq \delta \) expires, the birth rate \( \beta(a, \tau) \) can be set to 0. As a preliminary description, we will consider a policy-modified (truncated) birth rate function

\[ \beta(a, \tau) = \beta_0(a) \mathbb{1}(\tau, \delta) \] (6)

where the indicator function \( \mathbb{1}(\tau, \delta) = 1 \) for \( \tau > \delta \) and \( \mathbb{1}(\tau, \delta) = 0 \) for \( \tau \leq \delta \). This form assumes that once the imposed refractory period has past, the birth rate immediately rises back to a value associated with the persons current age.

### 2.2 Asymptotic behavior

We first analyze the asymptotic behavior of our model. An important feature of renewal transport equations such as the McKendrick model is that as \( t \to \infty \), the total population \( n(t) \) will grow exponentially (in the absence of nonlinear regulation terms [Gurtin and MacCamy, 1974]), while the normalized, age-dependent population density converges to a time-independent stationary distribution (see Perthame [2007, Chapter 3] and Arino [1995]). This property is independent of the initial condition. We will assume that this steady-state asymptotic property arises in
our two-component, three-variable model; i.e., the normalized densities \( f_0(t, a)/n(t) \) and \( f(t, a, \tau)/n(t) \) converge to stationary distributions. We denote the stationary limits as

\[
\lim_{t \to \infty} \frac{f_0(t, a)}{n(t)} = h_0(a) \quad \text{and} \quad \lim_{t \to \infty} \frac{f(t, a, \tau)}{n(t)} = h(a, \tau).
\]  

(7)

We also define the distribution associated with the total female population as

\[
\lim_{t \to \infty} \frac{f_{\text{tot}}(t, a)}{n(t)} = h_{\text{tot}}(a) = h_0(a) + \int_0^a h(a, \tau) d\tau,
\]  

(8)

where \( \int_0^\infty h_{\text{tot}}(a) da = 1 \). If we assume that \( f_0(0, a)/n(0) = h_0(a) \) and \( f(0, a, \tau)/n(0) = h(a, \tau) \) for any \( a, \tau \) at some initial time \( t = 0 \), then \( f_0(t, a)/n(t) = h_0(a) \) and \( f(t, a, \tau)/n(t) = h(a, \tau) \) hold for any \( t \geq 0 \).

From Eq. (4a), we have

\[
\frac{1}{f_0(t, a)} \frac{\partial f_0(t, a)}{\partial t} = - \frac{1}{f_0(t, a)} \frac{\partial f_0(t, a)}{\partial a} - \mu_t(a)
\]

\[
= - \frac{n(t)}{f_0(t, a)} \frac{1}{n(t)} \frac{\partial f_0(t, a)}{\partial a} - \mu_t(a)
\]

\[
= \frac{1}{h_0(a)} \frac{d h_0(a)}{d a} - \mu_t(a).
\]  

(9)

Thus, \( [\partial f_0(t, a)/\partial t]/f_0(t, a) \) is independent of \( t \). Moreover, for any \( a, a', \tau \)

\[
\frac{f_0(t, a)}{f(t, a', \tau)} = \frac{h_0(a)}{h(t, a', \tau)} = \frac{n(t)}{h(a', \tau)}
\]  

(10)

is also independent of \( t \) so that

\[
\frac{\partial}{\partial t} \left[ \frac{f_0(t, a)}{f(t, a', \tau)} \right] = \frac{1}{f(t, a', \tau)^2} \left[ f(t, a', \tau) \frac{\partial}{\partial t} f_0(t, a) - f_0(t, a) \frac{\partial}{\partial t} f(t, a', \tau) \right] = 0.
\]  

(11)

Thus, for any \( a, a', \tau \),

\[
\frac{1}{f_0(t, a)} \frac{\partial f_0(t, a)}{\partial t} = \frac{1}{f(t, a', \tau)} \frac{\partial f(t, a', \tau)}{\partial t} = \frac{1}{f_0(t, a')} \frac{\partial f_0(t, a')}{\partial t}.
\]  

(12)

Eqs. (9) and (12) show that \( [\partial f_0(t, a)/\partial t]/f_0(t, a) \) is independent of both \( t \) and \( a \), allowing us to define a constant that describes the stationary growth rate

\[
\lambda = \frac{1}{f_0(t, a)} \frac{\partial f_0(t, a)}{\partial t} = \frac{1}{f(t, a, \tau)} \frac{\partial f(t, a, \tau)}{\partial t} = \frac{1}{f_{\text{tot}}(t, a)} \frac{\partial f_{\text{tot}}(t, a)}{\partial t}.
\]  

(13)

Thus, we can express solutions for the densities \( f_0(t, a) \) and \( f(t, a, \tau) \) in the form

\[
f_0(t, a) = C h_0(a) e^{\lambda t} \quad \text{and} \quad f(t, a, \tau) = C h(a, \tau) e^{\lambda t},
\]  

(14)

where \( C \) is a constant. After using these expressions in Eqs. (4), we find the equations for the stationary distributions

\[
\frac{d}{da} h_0(a) = - (\mu_t(a) + \beta_0(a) + \lambda) h_0(a)
\]  

(15a)

\[
\frac{\partial}{\partial a} h(a, \tau) + \frac{\partial}{\partial \tau} h(a, \tau) = - (\mu_t(a) + \beta(a, \tau) + \lambda) h(a, \tau)
\]  

(15b)

\[
h_0(0) = \eta \left( \int_0^\infty \beta_0(a) h_0(a) da + \int_0^\infty da \int_0^a d\tau \beta(a, \tau) h(a, \tau) \right)
\]  

(15c)

\[
h(a, 0) = \beta_0(a) h_0(a) + \int_0^a \beta(a, \tau) h(a, \tau) d\tau.
\]  

(15d)
Next, using Eq. (5), we find
\[
\frac{d}{da} h_{\text{tot}}(a) = - (\mu_t(a) + \lambda) h_{\text{tot}}(a),
\]
which is solved by
\[
h_{\text{tot}}(a) = h_{\text{tot}}(0) \exp \left[ -a\lambda - \int_0^a \mu_t(a') da' \right].
\] (17)

We can then define the effective whole-population birth rate function
\[
\beta_{\text{eff}}(a) = \frac{\beta_0(a) h_0(a) + \int_0^a \beta(a, \tau) h(a, \tau) d\tau}{h_{\text{tot}}(a)},
\] (18)

which describes the overall birthrate weighted over the entire stationary population. This population-averaged birth rate \(\beta_{\text{eff}}(a)\) corresponds to that used in the basic lumped model (Eq. (1b)) and is the quantity that can be directly extracted from birth data that provide women’s ages at time of birth, but that may not distinguish whether females are first-time mothers. We prove in Appendix B that given \(\beta_{\text{eff}}(a)\), the new-mother birth rate function can be calculated from
\[
\beta_0(a) = \frac{\beta_{\text{eff}}(a)}{1 - \int_0^\infty \beta_{\text{eff}}(a - \tau) d\tau},
\] (19)

which then allows us to reconstruct \(\beta(a, \tau)\) from Eq. (6). Using \(\beta_{\text{eff}}\), the boundary condition for Eq. (16), the counterpart to Eq. (15c), can be written as
\[
h_{\text{tot}}(0) = \eta \int_0^\infty \beta_{\text{eff}}(a) h_{\text{tot}}(a) da.
\] (20)

Finally, after using the solution in Eq. 17 for \(h_{\text{tot}}(a)\) in Eq. (20), we find an equation for \(\lambda\):
\[
z(\lambda) \equiv \eta \int_0^\infty \beta_{\text{eff}}(a) \exp \left[ -a\lambda - \int_0^a \mu_t(a') da' \right] da = 1.
\] (21)

The function \(z(\lambda)\) is monotonically decreasing with \(\lambda\) and obeys the limits \(\lim_{\lambda \to +\infty} z(\lambda) = 0\) and \(\lim_{\lambda \to -\infty} z(\lambda) = +\infty\). Thus, Eq. (21) has a unique solution that can easily be found numerically. From Eq. (21), the solution for \(\lambda\)–the net population growth rate–clearly increases with \(\beta_{\text{eff}}(a)\) and decreases with \(\mu_t(a)\).

With \(\beta_0(a)\) and \(\lambda\) determined by Eqs. (19) and (21), respectively, we can explicitly find \(h(a, \tau)\). First, we use the normalization condition \(\int_0^\infty h_{\text{tot}}(a) da = 1\) on Eq. (16) to explicitly find \(h_{\text{tot}}(0)\) in terms of \(\lambda\) and \(\mu_t(a)\). Since \(h(0, \tau) = 0\), we have \(h_0(0) = h_{\text{tot}}(0)\), which allows us to explicitly express the solution to Eq. (15a):
\[
h_0(a) = h_0(0) \exp \left[ -a\lambda - \int_0^a (\mu_t(a') + \beta_0(a')) da' \right].
\] (22)

Next, we use Eq. (15d) and Eq. (18) to eliminate \(\beta(a, \tau)\) and find \(h(a, 0) = h_{\text{tot}}(a) \beta_{\text{eff}}(a)\), which is known. Thus, we can explicitly calculate \(h(a, \tau)\) by solving Eq. (15b) using the method of characteristics:
\[
h(a, \tau) = h(a - \tau, 0) \exp \left[ -\tau\lambda - \int_{a-\tau}^a (\mu_t(a') + \beta(a', a' - a + \tau)) da' \right].
\] (23)

To summarize, starting from \(\beta_{\text{eff}}(a)\), \(\mu_t(a)\), \(\eta\) (measured, say), and \(\delta\), we can compute the growth rate \(\lambda\) numerically, then analytically reconstruct \(\beta_0(a), \beta(a, \tau), h_0(a)\) and \(h(a, \tau)\).

We now use the Chinese national census data recorded in 1982 [Population Census Office under the State Council, 1985] to infer the overall birth rate \(\beta_{\text{eff}}(a)\) and the female death rate \(\mu_t(a)\) functions in China in 1981. Although gestation imposes a hard refractory period of \(\delta \approx 9\) months, some time is needed to recover from childbirth and the birth rate should more gradually recover. Specifically, in 1981 China, extended breastfeeding was common, which prevents the next pregnancy [Xu et al., 2009]. Thus, we will assume the birth rate returns to normal approximately only after about two years. Therefore, when there is no policy that controls the interval between births, we set \(\delta = 2\) years such that \(\beta(a, \tau) = 0\) for \(\tau < 2\) years and \(\beta(a, \tau) = \beta_0(a)\) for \(\tau > 2\) years. For other societies, this natural refractory period might be shorter. Using \(\delta = 2\) and Eq. (19), we calculate \(\beta_0(a)\) from \(\beta_{\text{eff}}(a)\) derived from
Figure 1: China’s 1981 birth rate and female death rate $\mu_f(a)$, calculated from 1982 national census data [Population Census Office under the State Council, 1985]. The red curve represents the observed birth rate $\beta_{\text{eff}}(a)$ for all women of a given age. The dashed blue curve represents the birth rate $\beta_0(a)$ for females who have not had any children. The area under the observed birth rate $\int_0^\infty \beta_{\text{eff}}(a)\,da$ represents the mean number of children born during over an individual’s lifetime.

These rates are illustrated in Fig. 1. Note that $\beta_0(a)$ for 1981 has already been mildly affected by the incipient birth-control policies in China.

Using the birth rate $\beta_{\text{eff}}(a)$ and death rate $\mu_f(a)$ shown in Fig. 1, we solve Eqs. (15) to find $h_0(a)$ and $h(a, \tau)$, and plot them with $h_{\text{tot}}(a)$ given by Eq. (17) in Fig. 2(a,b).

To explore the effects of an imposed refractory period, we first set $\delta = 2$ years, apply the newborn sex ratio of China in 1981, $\eta = 0.48$, and solve Eq. (21). We find $\lambda \simeq 0.005 > 0$, indicating an exponentially growing total population. This stationary growth rate is much smaller than the actual growth rate of China in 1981, which is 0.0146. One reason is that in 1981, the proportion of younger females is much higher than that in the stationary distribution $h_{\text{tot}}(a)$. The shape of $h_{\text{tot}}(a)$ is consistent with this growth as it is monotonically decreasing, indicating that every new generation has a larger population than the previous one. In Fig. 3, we see how increases in the refractory period $\delta$ decrease the asymptotic growth rate $\lambda$ and affect the distribution $h_{\text{tot}}(a)$. A negative overall birth rate $\lambda < 0$ (i.e., an asymptotically decaying population) arises when $\delta \gtrsim 3.22$ years $\approx 39$ months. As soon as $\lambda < 0$, the distribution $h_{\text{tot}}(a)$ becomes nonmonotonic, and a peak in the female population distribution arises at a finite age $a > 0$. 

Figure 2: The asymptotic population distribution associated with the birth and death rates shown in Fig. 1. (a) Steady-state age distributions of females without children, $h_0(a)$, and all females, $h_{\text{tot}}(a)$, respectively. A monotonically decreasing $h_{\text{tot}}(a)$ indicates that there are larger numbers of younger females, consistent with an increasing population and $\lambda > 0$. (b) The full double density $h(a, \tau)$ of females of age $a$ and whose youngest child is $\tau$ years old.
If \( \delta \) is set sufficiently large, a female cannot have a second child, and the outcome is equivalent to a strict one-child policy. We use the terminology “strict one-child policy” for the scenario in which each female can have strictly no more than one child, while we use “one-child policy” to refer to the actual policy realized in practice. From 1980 to 1990, the one-child policy contained many exceptions, allowing one to bear more than one child [Nie, 1999].

Figure 3: Asymptotic total-population growth rate and the steady-state, total-population age distribution. (a) Effect of an imposed interbirth delay \( \delta \) on the asymptotic growth rate \( \lambda \). Most of the decrease in \( \lambda \) occurs at small values of \( \delta \) where the most negative slopes arise. When \( \delta \) is large enough, a woman ages out before giving birth again, and imposing interbirth delay is equivalent to the strict one-child policy. (b) The steady-state age distribution \( h_{\text{tot}}(a) \). As \( \delta \) increases, the peak of population density moves from 0 to approximately 65.

Our formulation is valid only in the asymptotic case with a fixed delay \( \delta \) that remains unchanged for a long period of time. For practical modeling of policies in which delays \( \delta \) are used as a time-dependent control variable, such as China’s 1980 one-child policy and its subsequent modification in 2015, it is necessary to analyze the full model that delineates the two female populations.

2.3 Temporal evolution

As was used to predict the effects of the one-child policy, we use China’s female age distribution in 1981 [Population Census Office under the State Council, 1985] as a starting point to explore how the total population evolves under different values of the imposed delay \( \delta \). Since the data only contain total female numbers \( I_{\text{tot}}(a) \) and not \( I_0(a) \) and \( I(a, \tau) \) individually, we use \( I_0(a) = I_{\text{tot}}(a) h_{\text{tot}}(a)/h_{\text{tot}}(a) \) and \( I(a, \tau) = I_{\text{tot}}(a) h(a, \tau)/h_{\text{tot}}(a) \) to reconstruct these initial age distributions. These initial distributions are plotted in Fig. 4(a). With these initial conditions, we can solve Eq. (4a) and Eq. (4b) with the method of characteristics to find the full age and time dependence of the female populations

\[
f_0(t, a) = f_0(t - a, 0) \exp \left[ - \int_0^a \left( \mu(t) + \beta_0(a') \right) da' \right] \quad \text{if } t > a \tag{24a}
\]

\[
f_0(t, a) = I_0(a - t) \exp \left[ - \int_{a-t}^a \left( \mu(t) + \beta_0(a') \right) da' \right] \quad \text{if } t \leq a, \tag{24b}
\]

\[
f(t, a, \tau) = f(t - \tau, a - \tau, 0) \exp \left[ - \int_{a-\tau}^a \left( \mu(t) + \beta \left(a', a' - (a - \tau)\right) \right) da' \right] \quad \text{if } t > \tau \tag{25a}
\]

\[
f(t, a, \tau) = I(a - t, \tau - t) \exp \left[ - \int_{a-t}^{a-\tau} \left( \mu(t) + \beta \left(a', a' - (a - \tau)\right) \right) da' \right] \quad \text{if } t \leq \tau. \tag{25b}
\]

Females are fertile only between sexual maturity and menopause. Thus, we set \( a_{\min} \sim 12 \) years and \( a_{\max} \sim 50 \) years, so that \( \beta_0(a) = \beta(a, \tau) = 0 \) for \( a < a_{\min} \) or \( a > a_{\max} \). Recall that an imposed policy delayed-birth policy is
manifested by $\beta(a, \tau) = 0$ for $\tau < \delta$. Eq. (4c) becomes

$$f_0(t, 0) = \eta \left( \int_{a_{\min}}^{a_{\max}} \beta_0(a)f_0(t, a)da + \int_{a_{\min}}^{a_{\max}} da \int_{\delta}^{a} d\tau \beta(a, \tau)f(t, a, \tau) \right).$$  \hspace{1cm} (26)

For $t \leq \gamma \equiv \min\{a_{\min}, \delta\}$, the $f_0(t, a)$ and $f(t, a, \tau)$ terms in the integrands in Eq. (26) can be solved by Eq. (24b) and Eq. (25b). Thus, we can express $f_0(t, 0)$ in terms of $I_0(a)$, $I(a, \tau)$, $\beta_0(a)$, $\beta(a, \tau)$, and $\mu_\ell(a)$. Using Eqs. (24), we can calculate $f_0(t, a)$ for any $a$ and $t \leq \gamma$. Under the imposed refractory period, Eq. (4d) becomes

$$f(t, a, 0) = f_0(t, a)\beta_0(a) + \int_{\delta}^{a} \beta(a, \tau)f(t, a, \tau)d\tau.$$ \hspace{1cm} (27)

If $t \leq \gamma$, we can also use Eq. (25b) for $f(t, a, \tau)$ in the integrand of Eq. (27), and then use the solved $f_0(t, a)$ to express $f(t, a, 0)$ in terms of $I_0(a)$, $I(a, \tau)$, $\beta_0(a)$, $\beta(a, \tau)$, and $\mu_\ell(a)$. Using Eqs. (25), we can calculate $f(t, a, \tau)$ for any $a, \tau$ and $t \leq \gamma$. Finally, using $f_0(\gamma, a)$, $f(\gamma, a, \tau)$ as the initial conditions, we can solve $f_0(t, a)$, $f(t, a, \tau)$ for $t \leq 2\gamma$. Repeating this procedure, we can use $I_0(a)$, $I(a, \tau)$, $\beta_0(a)$, $\beta(a, \tau)$, and $\mu_\ell(a)$ to calculate $f_0(t, a)$ and $f(t, a, \tau)$ for any $t, a, \tau$.

Using the fundamental rates $\beta_0(a)$, $\mu_\ell(a)$ and $\beta(a, \tau)$ as those used in the previous subsection for the full model (see Fig. 1 and Eq. (6)), we use the above procedure to construct the total female population $n(t)$ (see Eq. 3). The evolution of $n(t)$ over one century, under different interbirth delays, are plotted in Fig. 4(b). At long times, the total population will decrease exponentially. Because the total population growth rate is most sensitive to small values of imposed delay $\delta$, even a delay of $\delta \sim 4$ years is sufficient to dramatically reduce population over the next 100 year, compared to the $\delta = 2$ case of no refractory period.

The results and analyses above can be generalized to include time dependent parameters $\mu_\ell(t, a)$, $\beta_0(t, a)$, $\beta(t, a, \tau)$, and even $\delta(t)$ to reflect social and policy changes. In this case, an imposed refractory period would be defined by the time-dependent birth rate function $\beta(t, a, \tau) = \beta_0(t, a)I(\tau > \delta(t))$ and the population densities will need to be evaluated numerically.

### 3 Results and Discussion

Our basic structured population model can be modified and applied to different scenarios and policies to make predictions about a number of potentially relevant quantities. We focus on the population dynamics in China under different control scenarios, paying particular attention to age and sex distributions.
3.1 Predictions and comparison to data from China

First, we use parameters inferred from 1981 Chinese data in our model to predict population growth for different values of $\delta$. When we compare predicted net growth rates with those derived from 1981-2020 data, shown in Fig. 5(a), we see that (i) during 1981-1990, the observed growth rate is close to that when $\delta = 2$; (ii) from 1991 to 2010, the observed growth rate was close to that predicted from a model with $\delta = 5$, possibly due to harsher policies like coerced abortion [Nie, 1999]; (iii) after 2011, the observed growth rate rose to a level close to that of a model with $\delta = 3 \sim 4$, indicating a de facto relaxation of the one-child policy. Indeed, after 2011, policies that encouraged births were initiated. Starting in 2011, a couple was allowed to have up to two children if both parents never had siblings. Then, starting in 2014, a couple can have up to two children if at least one of the parents never had a sibling. Finally, starting in 2016, couples could bear up to two children regardless of their sibling status [Kane and Li, 2021]. These policies might explain the flattening and subsequent increase in the net growth rate starting about 2011. However, the effects of these policies might be temporary. After the two-child policy in 2016, the net growth rate increased to the level consistent with a $\delta = 3$ model but soon returned to the level closer a $\delta = 4$ model. The actual growth rate is higher than in the $\delta = \infty$, strict one-child policy model. This indicates that many couples had, legally or illegally, more than one child.

Starting in 2021, a new policy allows any couple to have up to three children without penalty. We make different predictions for the effect of this three-child policy. The optimistic prediction is that the policy can fully stimulate childbearing to the point that the net growth rate can reach the level predicted by a $\delta = 2$ model, but this could be realized only if behavior and cultural-economic changes have not affected an intrinsic propensity for childbearing. Another possibility is that the policy has no significant effect so that the net growth rate will only increase modestly and transiently before effectively reducing back to that consistent with a $\delta = 4$ model. See Fig. 5(b) for different predictions of population over the 2021-2100 time frame.

3.2 Minimum childbearing age and population aging

Besides mandating a refractory period between births, another method of population control is to impose a minimum childbearing age. The minimum age $a_{\text{min}}$ is thus set by policy rather than by physiology. For example, starting in 1985, in Yicheng county, Shanxi province, a couple could bear two children, but the first child was allowed only after the mother turned 24, and the second child was allowed only after the mother turned 30 [Qin and Wang, 2017].

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1 This means that there was effectively no interbirth period policy and that the the adjusted birth rate $\beta_0(a)$ was not changed much during the lax birth-control policies during this period [Li and Wong, 2020].

2 In China, the legal marriage age for females is 20 implying an existing soft constraint of $a_{\text{min}} \approx 21$. 

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Figure 6: The effect of adjusting the minimum childbearing age. (a) Increasing the interbirth delay $\delta$ and increasing the minimum childbearing age $a_{\text{min}}$ have similar side effects of increasing the percentage of senior population. The red dashed line indicates a dangerous senior population percentage, 20%. (b) Under the strict one-child policy (namely, $\delta = \infty$), when we increase the minimum childbearing age $a_{\text{min}}$, the stationary net growth rate $\lambda$ will first increase then decrease.

One side-effect of population control is a distribution shifted toward older ages. In China, the percentage of seniors (65+) increased from 5% in 1981 to 13% in 2020 [Man et al., 2021]. Policies such as imposing interbirth delays $\delta$ and minimum childbearing ages $a_{\text{min}}$ can both affect the long term senior (65+) population. Fig. 6(a) shows a contour plot of the percentage of seniors as a function of imposed $\delta$ and $a_{\text{min}}$. In order to maintain the senior population under 20% (the red dashed line), the overall policy must not be too drastic. Nonetheless, with two strategies, a balanced combination can be used. For example, one can set $a_{\text{min}} = 26, \delta = 2$, or $a_{\text{min}} = 24, \delta = 4$, or $a_{\text{min}} = 21, \delta = 5$ and still prevent the senior population from exceeding 20% far in the future.

Although increasing the minimum childbearing age $a_{\text{min}}$ should reduce the rate of childbirth, we observe a counter-intuitive scenario in which the net growth rate is nonmonotonic in $a_{\text{min}}$. Under the strict one-child policy (i.e., $\delta = \infty$), increasing $a_{\text{min}}$ will first increase the stationary net growth rate $\lambda$ before decreasing it, as shown in Fig. 6(b). Under a perpetual strict one-child policy, the population in each successive generation is roughly halved and the total number of future newborns is roughly the current population. Although decreasing $a_{\text{min}}$ can temporarily increase the total population, it accelerates the “halving process” in the long run since the interval between successive generations is shorter. When $a_{\text{min}}$ is not large, almost every woman can have one child anyway. Further increasing $a_{\text{min}}$ will decrease $\lambda$ as more women start to be pushed past their childbearing years without giving birth; thus, a maximum in the growth rate $\lambda$ arises at $a_{\text{min}} \approx 29$.

3.3 Female population fraction and interbirth delay

We used $\eta = 0.48$, the fraction of female births 1981 China, to generate the results presented in Section 2. Due to subsequent sex-selective abortions biased towards males, the value of $\eta$ dropped to 0.45 in 2005 before gradually increasing [Zeng et al., 1992, Goodkind, 2011, Li and Meng, 2014].
Figure 7: Dependence of the stationary female fraction on $\eta$ and $\delta$. Increasing the fraction of female births $\eta$ and the interbirth delay $\delta$ can both increase the stationary female population fraction. The red dashed line indicates conditions for 50% females.

We can alter the value of $\eta$ and calculate the stationary female fraction of total population, which is also affected by the interbirth delay $\delta$. Fig. 7 illustrates the stationary female percentage as a function of $\eta$ and $\delta$. When $\delta = 2$, the stationary female percentage is approximately 1% higher than the female percentage $\eta$ at birth. When we fix $\eta$ and increase $\delta$, the stationary female fraction also increases. When $\delta = \infty$, the stationary female population is approximately 3% higher than $\eta$. For larger $\delta$, the stationary age distribution is shifted to larger ages. Since the female death rate $\mu_f(a)$ is lower than the male death rate $\mu_m(a)$ at larger ages $a$, the female percentage increases with age (in 1981 China, newborns were 48% female while 65+ seniors were 56% female).

3.4 Behavioral response to policies

We have discussed the policy of applying a refractory period between births and predicted its effects. We assume that the birth rate returns to normal after the refractory period, meaning that people obey this policy and do not respond with compensatory behaviors. In reality, people who want to have more children might mitigate the effects of control policies by, for example, giving birth again immediately after the end of the refractory period following the previous birth. In addition to this “catch up” strategy to recover from the “missed opportunity,” people might also prefer to have the first child earlier, so that the refractory period finishes at an younger age.

We propose a model that considers possible behavioral responses and compare it with a no-response model. (1) For females of age $a$ who have just finished their refractory period, the birth rate for the following year (only) will be set to $\beta_0(a)c_1$ instead of simply $\beta_0(a)$. We can model the compensatory increase of birth rate after the refractory period by $c_1 = 1 + 0.1 \times \min\{\delta - 2, 10\} > 1$. When the refractory period $\delta$ is longer, people are more likely to more quickly make up for the lost opportunity. (2) For females of age $a$ who have not had children, if $a + \delta \leq 40$, the birth rate, instead of simply being $\beta_0(a)$, will be set to $\beta_0(a)c_2$, where $c_2 = 1 + 0.05 \times \min\{\delta - 2, 10\}$. This means that females prefer to have the first child earlier, if they know that they are young enough to have another child after the refractory period (the age will be $a + \delta$ at that future time).
Figure 8: Comparison of the predictions of models with and without behavioral response. (a) Effect of an imposed interbirth delay $\delta$ on the asymptotic growth rate $\lambda$. The red curve depicts the no-response model, the same as the curve in Fig. 3(a). The blue curve is the growth rate associated with the behavioral response model (where $\beta_0(a)c_1$ and $\beta_0(a)c_2$ are used as birth rates). For a moderate refractory periods ($\delta = 5 \sim 10$), the behavioral response is equivalent to making $\delta$ one year shorter. (b) Evolution of the total female population $n(t)$ for different imposed delays between births. The red and blue curves represent populations associated with the no-response (same as in Fig. 4(b)) and behavioral response models, respectively. The solid, dashed, dotted, dash-dotted curves correspond to $\delta = 3, 4, 7, 15$ years, respectively. Behavioral responses have stronger effects on the total population when $\delta$ is not too large.

Fig. 8 compares predictions from the standard no-response model (red) to those from a behavioral response model (blue). As expected, behavioral responses blunt the policy-induced decreases in the stationary growth rate (Fig. 8(a)) and the total population (Fig. 8(b)), resulting in higher-than-expected growth and populations. For an imposed $\delta$, a compensatory behavioral response model leads to a higher stationary growth rate. In other words, if the behavioral responses of this example are included, the imposed delay $\delta$ would have to be about 1-2 years longer than in the absence of behavioral response in order to achieve the same overall growth rate (for intermediate delays $\delta \approx 5 - 15$ years). However, if the refractory period is set very long ($\delta \gtrsim 20$ years), our proposed behavioral responses are futile since females are irreversibly moved past their fertility window.

3.5 Comparison between China and Japan

We have examined the effect of applying a refractory period policy in China, which has implemented various birth-control policies over past four decades. To better study this interbirth delay, we apply our model to Japan, which does not have enforced polices on population control. We use Japan’s 2000 population data as a starting point [Statistics Bureau of Japan, 2001]. For Japan, we use its 2000 birth rate, which was much lower than that of 1981 China.

Fig. 9 compares the stationary growth rates between China and Japan, imposing different refractory periods $\delta$. Since Japan has a much lower growth rate $\beta_0$, with the same $\delta$, the stationary growth rate of Japan is lower than that of China. When $\delta$ is sufficiently large, since each female can have at most one child, the difference between China and Japan diminishes. In fact, the limiting high-$\delta$ stationary growth rate of Japan is slightly higher than that of China. Since the childbearing age is older in Japan, the gap between two successive generations is longer. As observed in Fig. 6(b), under large-$\delta$, sub-replacement conditions, a moderately longer gap between generations can increase the stationary (very long term) growth rate.
**4 Summary and Conclusions**

We have formulated a "continuum" of birth-control policies for population management in which the strict one-child policy is a limiting case. Our approach is based on explicitly incorporating a refractory period between births. In general, our age- and gestation-period structured model can also apply to organisms in which the gestation time is appreciable compared to an organism’s window of fertility. For example, animals such as the Greater cane rat (*Thryonomys swinderianus*), the Pacarana (*Dinomys branickii*) and the Steenbok (*Raphicerus campestris*) have gestation periods approximately 15% of their fertility window [Woods and Kilpatrick, 2005]. Although a single gestation period is about 3% of the childbearing period in humans [Tacutu et al., 2018], socially imposed refractory periods can be much longer (and infinite in a strict one-child policy). Thus, our model provides a natural way to test how imposed tunable interbirth refractory periods $\delta$ affect the predicted total female population and its steady-state age distribution. For long delays $\delta$, the model approaches the strict one-child policy as a larger fraction of women are pushed past menopause.

In our mathematical analysis, we found a number of analytic or closed-form solutions to relevant demographic quantities such as the steady-state age distribution. We then considered an alternative scenario in which "lax" birth control policies that was being implemented in 1981 are kept, along with an additional policy of an imposed refractory period between births. Using 1981 as the starting point, we predicted population levels and compared them to the actual, realized populations. By applying a refractory period $\delta$ between births and using 1981 China birth rates, we provided a retrospective analysis and arrived at a number of quantitative conclusions. Our analyses assumed that the birth rate $\beta_0$ and death rate $\mu$ did not change in the intervening years and that the population adhered to the birth-control policies without further behavioral responses. We concluded that: (i) When $\delta \gtrsim 3.2$ years, the total population will not grow in the long run (Fig. 3(a)); (ii) When $\delta \geq 4$ years, the total population in China would have always been maintained under 1.45 billion (Fig. 4(b)); (iii) When $\delta \geq 6$ years, the net growth rates during 1990-2010 (when a harsher one-child policy was applied) would be as low as what was realized (Fig. 5(a)); (iv) Without increasing the minimum childbearing age, when $\delta \leq 5$ years, the stationary senior population would be maintained under 20% (Fig. 6(a)).

Such predictions assume the adjusted birth rate $\beta_0(a)$ does not change over time. This assumption is definitely unrealistic, since many important socio-economic factors can affect birth rate distributions. The decrease of birth rate in China (illustrated in Fig. 4(a)) is not due solely to birth-control policies. After 1980, female education increased, which had the statistically significant effect of decreasing the birth rate [Lan and Kuang, 2016]. Additional evidence consistent with an extra-policy influence on birth rates in China is the increase in the average age of first childbirth.
(which one expects to be less affected by policies) from 24.3 years to 26.9 years from 2006 to 2016 [He et al., 2019]. Moreover, we expect behavioral responses to policies that could mitigate their effectiveness. Since it is difficult to separate and quantify the effects of socio-economic factors and behavioral responses on birth rates, we did not explicitly incorporate these factors in our model. Nonetheless, we discussed how policies can be implemented through different modifications of age- and refractory period-dependent birth rate functions. For example, we considered a population-control policy whereby a minimum birth age $a_{\text{min}}$ is imposed. Here, we found the counterintuitive result that under a strict one-child policy, increasing $a_{\text{min}}$ first increases the stationary net growth rate, before decreasing it as $a_{\text{min}}$ is further increased.

Age-structured models can also be generalized to include additional subpopulations, such as those arising in cell division [Xia et al., 2020] and disease propagation [Kang and Ruan, 2021] models. For example, in the birth control context, different generations and family structure can be enumerated in order to predict the effects of policies such as those implemented in 2011 and 2014 that consider the sibling status of would-be parents, allowing those without siblings more latitude in childbirth. Additional concepts from sociology and response to socioeconomic and political influences can also potentially be integrated for a more complete framework of population dynamics and demography. The ideas and mathematical tools in this paper can be adapted to other fields. For example, an economist or a sociologist might study the cultural norms regarding child spacing and use our models to connect child spacing to growth rates.

**Data accessibility**

Data and relevant code for this research work are stored in GitHub: https://github.com/YueWangMathbio/ChildPolicy and have been archived within the Zenodo repository: https://doi.org/10.5281/zenodo.6394805.

**Authors’ contributions**

All authors collected and reviewed the literature, developed the model and analysis, and wrote drafts of the manuscript. YW and RD analyzed data and developed the computational methodology. All authors contributed to visualization and editing of the manuscript. All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

**Competing interests**

We declare we have no competing interests.

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**Mathematical Appendices**

A **The equation of $f_{\text{tot}}(t, a)$**

We show by direct substitution that the total female population density $f_{\text{tot}}(t, a)$ satisfies the standard age-structured McKendrick equation. Substitution of Eq. 2 into Eq. (5) and expanding,
\[
\frac{\partial}{\partial t} f_{\text{tot}}(t, a) + \frac{\partial}{\partial a} f_{\text{tot}}(t, a) = \frac{\partial}{\partial t} f_0(t, a) + \frac{\partial}{\partial a} f_0(t, a) + \int_0^a \frac{\partial}{\partial t} f(t, a, \tau) d\tau + \int_0^a \frac{\partial}{\partial a} f(t, a, \tau) d\tau + f(t, a, a) + \int_0^a \frac{\partial}{\partial \tau} f(t, a, \tau) d\tau - \int_0^a \frac{\partial}{\partial \tau} f(t, a, \tau) d\tau = - \mu_t(a) f_0(t, a) - \int_0^a \mu_t(a) f(t, a, \tau) d\tau + f(t, a, a) - f(t, a, a) + f(t, a, 0) = - \mu_t(a) f_{\text{tot}}(t, a) - \beta_0(a) f_0(t, a) - \int_0^a \beta(a, \tau) f(t, a, \tau) d\tau + \beta_0(a) f_0(t, a) + \int_0^a \beta(a, \tau) f(t, a, \tau) d\tau = - \mu_t(a) f_{\text{tot}}(t, a). \tag{28}
\]

\[h(a, \tau) = \exp \left[ -\tau \lambda - \int_{a-\tau}^a \mu_t(a') da' \right], \tag{29}\]

while from Eq. (17), we have
\[\frac{h_{\text{tot}}(a)}{h_{\text{tot}}(a-\tau)} = \exp \left[ -\tau \lambda - \int_{a-\tau}^a \mu_t(a') da' \right]. \tag{30}\]

Thus,
\[\frac{h_{\text{tot}}(a)}{h_{\text{tot}}(a-\tau)} = \frac{h(a, \tau)}{h(a-\tau, 0)}. \tag{31}\]

From Eq. (15d) and Eq. (18), we have
\[h(a-\tau, 0) = \beta_{\text{eff}}(a-\tau) h_{\text{tot}}(a-\tau). \tag{32}\]

Upon combining Eq. (31) and Eq. (32), we arrive at
\[h(a, \tau) = \beta_{\text{eff}}(a-\tau) h_{\text{tot}}(a). \tag{33}\]

Since Eq. (33) is valid for any \(\tau \leq \delta\), we have
\[\int_0^\delta \beta_{\text{eff}}(a-\tau) d\tau = \frac{\int_0^\delta h(a, \tau) d\tau}{h_{\text{tot}}(a)}. \tag{34}\]

Eq. (18) can be transformed into
\[\beta_{\text{eff}}(a) = \beta_0(a) \frac{h_0(a) + \int_0^a h(a, \tau) d\tau}{h_{\text{tot}}(a)} = \beta_0(a) \left[ 1 - \frac{\int_0^\delta h(a, \tau) d\tau}{h_{\text{tot}}(a)} \right]. \tag{35}\]

Combining Eq. (34) and Eq. (35), we obtain \(\beta_{\text{eff}}(a) = \beta_0(a) \left[ 1 - \int_0^\delta \beta_{\text{eff}}(a-\tau) d\tau \right]\) and thus
\[\beta_0(a) = \frac{\beta_{\text{eff}}(a)}{1 - \int_0^\delta \beta_{\text{eff}}(a-\tau) d\tau}. \tag{36}\]
References


