Math 117, Winter 00: Homework 1

Remark. The following symbols are used:
$\forall$ means "for all". For example, $\forall n > 2$ means "for all n greater than 2".
$\mathbb{N}$ is the set of natural numbers, $\{1, 2, 3, ...\}$.
$\mathbb{Z}$ is the set of integers, $\{0, \pm 1, \pm 2, ...\}$.
$\in$ means "is an element of", so for example $n \in \mathbb{N}$ means "n is an element of $\mathbb{N}$", i.e. "n is a natural number".

2A/E1. Prove that $\forall n \geq 4 : n! \geq 2^n$.

Proof. We have to start our induction at $n = 4$. Since $4! = 24$ and $2^4 = 16$, $P(4)$ holds. Suppose $P(n)$ holds. Then

$$(n+1)! = (n+1) \times n! \geq (n+1) \times 2^n > 2 \times 2^n = 2^{n+1}.$$

Thus, $P(n+1)$ also holds. Here the first inequality is the induction assumption and the second inequality follows since $n \geq 4$, so certainly $(n+1) > 2$.

2A/E4. (i) Prove that $\forall n \geq 1$:

$$\frac{1}{1-x} = 1 + x + ... + x^{n-1} + \frac{x^n}{1-x}.$$

(ii) Substituting $x = 2$ in (i), prove that $\forall n \geq 1$:

$$1 + 2 + ... + 2^{n-1} = 2^n - 1.$$

Proof. (i) For $n = 1$, the statement is that $\frac{1}{1-x} = 1 + \frac{x}{1-x}$, which is true. Suppose the statement holds for n. Then

$$1 + x + ... + x^{n-1} + x^n + \frac{x^{n+1}}{1-x} = 1 + x + ... + x^{n-1} + \frac{x^n - x^{n+1}}{1-x} + \frac{x^{n+1}}{1-x}$$

$$= 1 + x + ... + x^{n-1} + \frac{x^n}{1-x} = \frac{1}{1-x}.$$

(ii) We could do this directly by induction, but it is faster and easier to substitute $x = 2$ in (i). We get

$$\frac{1}{1-2} = 1 + 2 + ... + 2^{n-1} + \frac{2^n}{1-2}$$

$$\Rightarrow -1 = 1 + 2 + ... + 2^{n-1} - 2^n$$

$$\Rightarrow 2^n - 1 = 1 + 2 + ... + 2^{n-1}.$$

2A/E10. The Tower of Hanoi

Here is the main idea of this problem: if you know how many moves you need to move n disks from one pole to another, then you can get the number of moves for $n+1$ disks from that as follows.

a) move the top n disks to the middle pole using the third pole as the "spare".

b) move the $(n+1)^{st}$ disk from the first pole to the third pole.

c) move the n disks from the middle pole to the third pole, using the first one as the spare this time.

So how many moves do you need? For 1 disk, you obviously need 1 move. If for n disks you need k moves, then by the above explanation you need $2k+1$ moves for $n+1$ disks (k for a), 1 for b), k for c)). So for two disks you need $2+1=3$, for three $2\times 3+1=7$, for four $2\times 7+1=15$ and so on. Looking at this pattern, we claim:

The number of moves needed to move n disks is $2^n - 1$.

Proof. For 1 disk we need 1 move, so the statement is true for $n=1$.

Suppose now the statement holds for n disks. For $n+1$ disks, we first move n disks, then one disk, and then n disks. The total number of moves is thus

$$(2^n-1)+1+(2^n-1) = 2\times 2^n - 1 = 2^{n+1}-1.$$

Question: How many moves does it take to move n disks if there are four poles? Five? Is there a general formula?

2A/E14. What's wrong with the Baby Theorem?

When you do induction, you can only use the induction assumption on the **first** n elements. In the proof in the book, the set R of the n babies to the right includes the elements 2 through $n+1$, and we can't use our induction assumption on the $n+1^{st}$ baby. **Don't assume what you're trying to prove!!**

Alternatively, you could just say that in the case $n=2$, there is no overlap, so you can't compare the set L and the set R.

2B/E4. Prove that the sum of the interior angles of an n-sided convex polygon is $180° \times (n-2)$.

Proof. It obviously doesn't make sense to talk about an n-sided convex polygon if $n < 3$, so we'll start at $n = 3$. In this case, we have a triangle, and the sum of the interior angles is $180° = 180° \times (3-2)$. So the statement holds for $n = 3$.

If it holds for all n-sided convex polygons, let's consider one with $n+1$ sides (and vertices). Now take any three adjacent vertices and connect the outer two with a line. What you get is an n-sided polygon and a triangle. Also, to add up all the interior angles of the $n+1$-sided polygon is the same as to add up all the interior angles of the n-sided polygon and the triangle. Thus, the sum of the interior angles of the $n+1$-sided polygon is

$$180° \times (n-2) + 180° = 180° \times (n-1).$$

19A/E1. Prove that the sum of the elements in the nth row of Pascal's triangle is 2^n for each n. How many subsets of a set with n elements are there?

Proof. The nth row of Pascal's triangle consists of $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$. By the Binomial theorem,

$$2^n = (1+1)^n = \binom{n}{0} + \ldots + \binom{n}{n}.$$

By a theorem in the book (on page 294), if S is a set with n elements, then it has $\binom{n}{r}$ subsets with r elements. Since all sizes between 0 and n elements are possible for the subsets (0 would be the empty set and n all of S), the total number of subsets is the sum we just calculated, 2^n. Try to prove this directly by induction on the number n of elements of the set (not too hard).