Math 3228 - Week 7

- Winding numbers
- The argument principle
- Rouche's theorem
- The fundamental theorem of algebra revisited

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Winding numbers

- In this course you have already seen the *Residue theorem*, which is as follows:

- **Residue theorem, first version.** Let γ be a simple closed curve traversed once anti-clockwise, and let D be the domain enclosed by γ. Let f be a function which is analytic on $\gamma \cup D$ except at a finite number of singularities $z_1, \ldots, z_n$ in D. THen we have

$$\int_\gamma f(z)\ dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}_{z=z_k} f(z).$$

- **Example.** Take $f(z) = 1/z$, this is has a simple pole at 0 with residue 1 (why?), and is analytic everywhere else in the complex plane $\mathbf{C}$. So if γ is a simple closed curve traversed once anti-clockwise that encloses the origin, e.g. $\gamma(t) = e^{it}$ for $0 \leq t \leq 2\pi$, then $\int_\gamma f(z)\ dz = 2\pi i$ (one can of course also obtain this from the Cauchy integral formula, which is a special case of the Residue theorem). If however γ is a simple closed curve that does not enclose the origin, e.g. $\gamma(t) = 3 + e^{it}$ for $0 \leq t \leq 2\pi$ then $f(z)$ is analytic on and inside γ, and then by Cauchy's theorem $\int_\gamma f(z)\ dz = 0$. If instead γ is a simple closed curve traversed *clockwise* that encloses the origin, e.g. $\gamma(t) = e^{-it}$ for $0 \leq t \leq 2\pi$, then we have instead $\int_\gamma f(z)\ dz = -1$, since the integral along γ is the negative of the integral along $-\gamma$ (the reversal of γ). Finally, if γ is a non-simple closed curve which encloses the origin twice, e.g. $\gamma(t) = e^{it}$ for $0 \leq t \leq 4\pi$, then we have $\int_\gamma f(z)\ dz = 4\pi i$, because we can break γ into the sum of two smaller simple closed curves, each of which enclose the origin. Note that in each of these cases γ does not actually pass through the singularity, but only goes around it. If γ passed through the singularity then the integral would only make sense as a principal value integral, as the integrand would be going to infinity at a point on γ.

- These examples show that in order to compute an integral $\int_\gamma f(z)\ dz$ when γ is closed and f has a finite number of singularities, it is not always enough to simply just add up all the residues of f inside γ and multiply by $2\pi i$; the winding number of γ around the singularities makes a difference. We can quantify this by the following generalized residue theorem.

- **Residue theorem, second version.** Let D be a simply connected domain, and f be a function analytic on D except at a finite number of singularities $z_1, z_2, \ldots, z_n$. Let γ be a closed (but not necessarily simple) curve in D which does not pass through any of the singularities $z_1, \ldots, z_n$. Then we have

$$\int_\gamma f(z)\ dz = 2\pi i \sum_{k=1}^{n} W(\gamma, z_k) \mathrm{Res}_{z=z_k} f(z);$$

in other words, we add up all the residues as before, but now we multiply each residue by the winding number of γ around z_k (which could be a positive integer, negative integer, or zero).

- Recall from previous lectures that the winding number of a closed curve γ around a point z_0 which does not lie on γ is defined by

$$W(\gamma, z_0) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}.$$

The winding number is not defined if z_0 passes through γ. The winding number is always an integer, and (informally speaking) counts how many times γ winds anti-clockwise around z_0. It is additive: if γ_1, γ_2 are two closed curves with the same starting point, then $W(\gamma_1 + \gamma_2, z_0) = W(\gamma_1, z_0) + W(\gamma_2, z_0)$. Similarly we have $W(-\gamma, z_0) = -W(\gamma, z_0)$.

- **Proof of second version of residue theorem.** Write $a_k := \mathrm{Res}_{z=z_k} f(z)$, i.e. a_k is the residue of f at z_k. We construct the auxiliary function g as

$$g(z) := \sum_{k=1}^{n} \frac{a_k}{z - z_k}.$$

Observe that g is analytic everywhere except at $z_1, \ldots, z_n$ and has simple poles at each of these points. Furthermore, the residue of g at z_k is just a_k. For instance, near z_1, g is the sum of $\frac{a_1}{z-z_1}$ (which clearly has a residue of a_1 at z_1), plus the other terms $\sum_{k=2}^{n} \frac{a_k}{z-z_k}$, which are analytic in a neighbourhood of z_1 and hence do not contribute to the residue there. Now we split

$$\int_\gamma f(z)\ dz = \int_\gamma g(z)\ dz + \int_\gamma (f(z) - g(z))\ dz$$

and consider the two integrals separately. (This is the trick of "adding and subtracting" an auxiliary term, and is extremely powerful in analysis - provided, of course, that you choose a sufficiently useful auxiliary term).

- First consider $\int_\gamma g(z)\ dz$. We can expand this as

$$\int_\gamma g(z)\ dz = \int_\gamma \sum_{k=1}^{n} \frac{a_k}{z - z_k}\ dz = \sum_{k=1}^{n} a_k \int_\gamma \frac{dz}{z - z_k}$$

since we can interchange integrals and finite sums. (For infinite sums, one requires uniform convergence of the summands, e.g. by use of the "Weierstrass M-test"). But by definition of the winding number we thus have

$$\int_\gamma g(z)\ dz = \sum_{k=1}^{n} a_k W(\gamma, z_k) = \sum_{k=1}^{n} W(\gamma, z_k) \mathrm{Res}_{z=z_k} f(z)$$

which was what we wanted. So now all we have to do is to make sure the integral $\int_\gamma f(z) - g(z)\ dz$ vanishes.

- Since f and g have the same set of singularities $z_1, \ldots, z_n$, and their residues match (they both have residue a_k at z_k), we see that $f - g$ is analytic everywhere except at $z_1, \ldots, z_n$, and that the residue of $f - g$ is zero at every one of these singularities. (This does not quite mean that the singularities are removable, though. Only the $\frac{1}{z-z_k}$ term in the Laurent expansion of $f - g$ around z_k is guaranteed to vanish; the higher order terms such as $\frac{1}{(z-z_k)^2}$ might still be present).

- By the first residue theorem, we thus know that $\int_{\gamma'} f(z) - g(z)\, dz = 0$ whenever γ' is any simple closed curve in D traversed once anticlockwise that does not pass through $z_1, \ldots, z_k$. The same is then true for simple closed curves traversed clockwise, since $-0 = 0$. The same is then true for any closed curve in D that does not pass through $z_1, \ldots, z_k$, since one can break up any closed curve into a union of simple closed curves. In particular, we have $\int_\gamma f(z) - g(z)\, dz = 0$ as desired. Combining this with our previous computation of $\int_\gamma g(z)\, dz$ we obtain the second form of the residue theorem $\square$

- So once we know where all the singularities of an analytic function are, and what the residues are at each singularity, one can then easily integrate that function on any closed curve that doesn't pass through the singularities, just by counting winding numbers. (Integration on open curves is still difficult, however; while contour integration is a wonderful and powerful tool, it isn't a magic wand and can't integrate just anything. This is why we still teach real integration techniques in addition to contour integration).

- For a very complicated closed curve γ, one way to compute the winding number around a point z_0 is as follows. Draw a ray r from z_0 to infinity in any direction. Count the number of times γ crosses r from the clockwise side of r to the anticlockwise side of r, and subtract the number of times it crosses from the anticlockwise to the clockwise. (If γ becomes tangent to r but does not actually cross r, do not count this as a crossing). This subtraction will give you the winding number. For instance, using the negative real axis from 0, the winding number of a closed curve γ is equal to the number of times γ crosses from the second quadrant $\{x + iy : x < 0 < y\}$ to the third quadrant $\{x + iy : x, y < 0\}$, minus the number of times it crosses back from the third quadrant to the second. (This is closely related to the Log function, which is an antiderivative of $1/z$ which jumps up by $2\pi i$ when crossing from the second quadrant to the third, and jumps down by $-2\pi i$ when crossing from the third back to the second. Can you see how this is related to the winding number $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z}$ of γ around 0? Use the fundamental theorem of calculus.)

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The argument principle

- We apply the above version of the residue theorem to count the zeroes and poles of a meromorphic function.

- **Definition.** Let D be a domain. We say that a function $f : D \to \mathbf{C}$ is *meromorphic on* D if it only has a finite number $z_1, \ldots, z_n$ of singularities in D, and all of those singularities are poles or removable singularities (i.e. no essential singularities; all Laurent series around a singularity have only a finite number of singular terms). A meromrphic function with no singularities at all is called *analytic* or *holomorphic*.

- **Example.** In the disk $|z| < 1$, the function $\frac{1}{z(z-2)(z-1/2)}$ is meromorphic, with poles at 0 and 1/2 (the pole at 2 is irrelevant since it is outside the domain of interest). But the function $e^{1/z}$ is not meromorphic because it has an essential singularity at 0. (However, the function $e^{1/(z-2)}$ is meromorphic on the disk, indeed it is holomorphic here.

- Meromorphic functions of course have a residue at every pole, but it is sometimes difficult to compute. But if instead of considering a meromorphic function f, one considers another function called the *logarithmic derivative* of f, the residues become very easy (and useful) to compute.

- **Definition** Let f be a meromorphic function on a domain D with only finitely many zeroes. We define the *logarithmic derivative* of f to be the function $\frac{f'(z)}{f(z)}$.

- Th logarithmic derivative is not defined when f has a singularity (clearly), but is also not defined when f has a zero (since then we are dividing by zero). Everywhere else, though, the logarithmic derivative is defined and is even analytic (because we are dividing one analytic function by another non-zero analytic function). The reason why $\frac{f'(z)}{f(z)}$ is called logarithmic derivative is because, formally, it is what the chain rule would say the derivative of $\log f(z)$ is. (Unfortunately $\log f(z)$ is multi-valued and so this isn't quite correct, but it is fairly close to accurate and so calling this the logarithmic derivative is not a bad idea.)

- **Examples.** The logarithmic derivative of e^{z^2} is $2z$. The logarithmic derivative of $z/(z-1)$ is $\frac{1}{z} - \frac{1}{z-1}$.

- **Exercise** Let f, g be meromorphic functions on D with only finitely many zeroes. Show that the logarithmic derivative of fg is the sum of the logarithmic derivatives of f and g separately, while the logarithmic derivative of f/g is the difference of the logarithmic derivatives of f and g separately. Notice how this gives a very nice way to phrase the product and quotient rules for differentiation. It also reinforces why we call this the *logarithmic* derivative, since it turns products and quotients into sums and differences.

- Now we compute the residues of the logarithmic derivative.

- **Theorem.** Let f be a meromorphic function on a domain D with only finitely many zeroes. If f has a pole at z_0 with order m, then the logarithmic derivative $\frac{f'(z)}{f(z)}$ has a simple pole at z_0 with residue $-m$. If instead f has a zero at z_0 with order m, then the logarithmic derivative $\frac{f'(z)}{f(z)}$ has a simple pole at z_0 with residue m.

- **Proof.** First suppose that f has a pole of order m at z_0. Then in a neighbourhood of z_0 we have a Laurent expansion

$$f(z) = a_{-m}(z - z_0)^{-m} + a_{-m+1}(z - z_0)^{-m+1} + \ldots$$

with the first co-efficient a_{-m} non-zero. We can factorize this as

$$f(z) = (z - z_0)^{-m}(a_{-m} + a_{-m+1}(z - z_0) + \ldots).$$

Meanwhile, if we differentiate the Laurent series term-by-term (this is rigorous, as was shown in previous lectures) we have

$$f'(z) = -ma_{-m}(z - z_0)^{-m-1} + (-m+1)a_{-m+1}(z - z_0)^{-m} + \ldots$$

which can be factorized as

$$f'(z) = (z - z_0)^{-m-1}(-ma_{-m} + (-m+1)a_{-m+1}(z - z_0) + \ldots)$$

and thus the logarithmic derivative takes the form

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_0} \frac{-ma_{-m} + (-m+1)a_{-m+1}(z - z_0) + \ldots}{a_{-m} + a_{-m+1}(z - z_0) + \ldots}.$$

Now look at the second quotient on the right-hand side as z approaches z_0. The numerator approaches $-ma_{-m}$, and the denominator approaches a_{-m}, both of which are non-zero. Thus the second quotient is the quotient of two non-zero analytic functions, and its limiting value at z_0 is $-m$. Thus we have a Taylor expansion

$$\frac{-ma_{-m} + (-m+1)a_{-m+1}(z - z_0) + \ldots}{a_{-m} + a_{-m+1}(z - z_0) + \ldots} = -m + b_1(z - z_0)b_2(z - z_0)^2 + \ldots$$

and thus the logarithmic derivative has the Laurent expansion

$$\frac{f'(z)}{f(z)} = \frac{-m}{z - z_0} + b_1 + b_2(z - z_0)^2 \ldots$$

around z_0. In particular, the logarithmic derivative has a simple pole at z_0 with residue $-m$ as claimed.

- The case when f has a zero of order m at z_0 is very similar and is left to the reader. (In fact, poles and zeroes are opposites of each other in many ways; one can think of a pole of order m as a zero of order $-m$, or conversely a zero of order m as a pole of order $-m$). □

- **Example** The function $f = (z-2)^3 e^z/(z-1)^4$ has a triple zero at 2 (why?), and hence its logarithmic derivative f'/f has a simple pole at 2 with residue 3. Similarly it has a simple pole at 1 with residue -4 (why?).

- Combining the above theorem with the residue theorem, we obtain

- **Corollary.** Let D be a simply connected domain, and let f be a meromorphic function on D with finitely many zeroes and no removable singularities (i.e. the only singularities are poles). Then for any simple closed curve γ traversed once anticlockwise which does not pass through any zero or pole of f, the quantity

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\ dz$$

is equal to the number of zeroes inside γ (counted with multiplicity, e.g. a double zero counts twice), minus the number of poles inside γ (also counted with multiplicity). In particular, the above quantity must be an integer.

- **Example.** Consider the function $f = (z-2)^3 e^z/(z-1)^4$ mentioned earlier; this has a triple zero at 2, a quadruple pole at 1, and no other zeroes or poles. Thus if γ is the circle $\gamma(t) = 3e^{it} : 0 \leq t \leq 2\pi$, then $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\ dz = 3 - 4 = -1$. If γ' is the circle $\gamma(t) = \frac{3}{2}e^{it} : 0 \leq t \leq 2\pi$, then $\frac{1}{2\pi i}\int_{\gamma'} \frac{f'(z)}{f(z)}\ dz = -4$.

- This is already a useful formula - it means that one can count the number of zeroes and poles of a meromorphic function f inside a region by integrating its logarithmic derivative on the boundary of that region. More precisely, the formula does not count the number of zeroes or the number of poles separately, but rather counts the difference between the two. This is a fact of life; if a function has both zeroes and poles it is difficult to find a way to just look at the poles or just look at the zeroes. For instance, consider the function $\frac{z}{z-0.0001}$ in the disk $\{z : |z| < 1\}$. This function is meromorphic with one zero at 0 and one pole at 0.0001. However, the zero and the pole almost cancel each other out and if one was measuring this function on the boundary of the disk one would be hard pressed to distinguish this function from just the constant function 1, which has no zeroes and no poles. (There is of course some effect of the combined zero-pole pair - it makes the function slightly larger than 1 on the right half of the unit circle, and slightly less than 1 on the left half, but this "dipole" effect is much weaker than what a single pole (such as $\frac{1}{z-0.0001}$) or single zero (such as z) would do to the function on the unit circle). However, in many cases one knows in advance that there are no poles (e.g. f might be a polynomial) and then this formula gives a way to count how many zeroes a function has inside any given curve. This formula is especially useful for computer algorithms to count zeroes, because we know how to teach a computer to compute integrals numerically. And the algorithm is robust; if the computer makes a roundoff error and computes $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$ as 3.99998, we can see what happened and work out that there were really four zeroes inside γ (or at least that there are four more zeroes than poles).

- Note also that one has to count multiplicity properly; a function such as z^2 with one double zero is very close to $z(z-0.00001)$ which has two simple zeroes. It would be very difficult to design a zero-counting formula that could distinguish these two functions, so it makes sense to count a double zero as if it were two simple zeroes.

- There is another way to view the formula. Suppose we make the change of variables
$$w = f(z)$$
and hence $dw = f'(z)\,dz$. Since z traverses the curve γ, w will therefore traverse the curve $f(\gamma)$, the image of γ under f; this new curve is parameterized by the function $f(\gamma(t))$, where the time parameter t ranges over the same interval as with the original curve. By the change of variables formula (which is just as valid for complex integrals as it is for real integrals, and it has the same proof) we thus have
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_{f(\gamma)} \frac{dw}{w}.$$
But the right-hand side is nothing more than the winding number of $f(\gamma)$ around zero. We have thus proven

- **Argument principle.** Let D be a simply connected domain, and let f be a meromorphic function in D with finitely many zeroes and no removable singularities. Then for any simple closed curve γ traversed once anti-clockwise which avoids all the poles and zeroes of f, the number of zeroes of f inside γ (counting multiplicity), minus the number of poles of f (counting multiplicity), is equal to the winding number of $f(\gamma)$ around the origin.

- **Example.** Consider the function $f(z) := z^5$, and let γ be the curve $\gamma(t) = e^{it} : 0 \leq t \leq 2\pi$. Then f has a quintuple zero at 0, and thus has five zeroes and no poles inside γ. Meanwhile, $f(\gamma)$ is the curve $f(\gamma(t)) = e^{5it} : 0 \leq t \leq 2\pi$, which winds five times anti-clockwise around the origin. Since $5 - 0 = 5$, we see that this is consistent with the argument principle. Now suppose we replace f by the function $g(z) = z^{-5}$, which has five poles inside γ and no zeroes. Now $g(\gamma)$ is the curve $g(\gamma(t)) = e^{-5it} : 0 \leq t \leq 2\pi$, which winds five times *clockwise* around the origin. Since $0 - 5 = -5$, this is again consistent with the argument principle.

- Informally, what the argument principle is telling us is that every zero of a meromorphic function f twists the complex plane around the origin once anti-clockwise, whereas every pole twists the complex plane around the origin once clockwise. The total twisting around the origin of a curve γ by the function f - i.e. the winding number of $f(\gamma)$ around the origin - is thus equal to the number of zeroes inside γ minus the number of poles. A double zero will twist twice as much as a simple zero, and so forth.

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Rouche's theorem

- The argument principle has the following useful consequence: if one perturbs a complex function by a small amount, then the number of zeroes minus poles that it contains does not change.

- **Rouche's theorem.** Let γ be a simple closed curve enclosing a domain D, and let f, g be meromorphic functions on $\gamma \cup D$ which have finitely many zeroes, no removable singularities, and no poles on γ (i.e. they are all inside D). Suppose also that

$$|g(z) - f(z)| < |f(z)| \text{ for all } z \in \gamma;$$

i.e. at every point z of the curve γ, $g(z)$ is closer to $f(z)$ than the origin is. Then $f(\gamma)$ and $g(\gamma)$ have the same winding number around the origin, and thus (by the argument principle) the number of zeroes minus poles of f in D is equal to the number of zeroes minus poles of g.

- This theorem is occasionally called the "Walking the dog" theorem; it says that if you (at $f(z)$) are walking a dog (at $g(z)$) around a lamp-post (at 0), and you always keep the length of the leash ($|f(z)-g(z)|$) between you and the dog shorter than the distance ($|f(z)|$) to the lamp-post, then you and the dog always have the same winding number, i.e. the leash cannot get tangled up in the lamp-post.

- **Proof.** Let z be any point in γ. By hypothesis we have $|f(z)| > |g(z)-f(z)|$, in particular $|f(z)| > 0$, i.e. $f(z)$ is non-zero. Also $g(z)$ is non-zero, since if $g(z)$ were zero then $|f(z)|$ would equal $|g(z)-f(z)|$, a contradiction. Since f is non-zero on γ, and we can define the function $h(z)$ by $h(z) := g(z)/f(z)$, which will be analytic everywhere on γ and also meromorphic on D. Dividing the hypothesis $|g(z)-f(z)| < |f(z)|$ by $|f(z)|$ we obtain that $|h(z)-1| < 1$, i.e. for every $z \in \gamma$, $h(z)$ is contained in the unit ball of radius 1 centered at 1. In particular, $h(z)$ is never on the negative real axis, and thus $\mathrm{Log} h(z)$ is analytic on γ (where Log is the principal branch of the logarithm. This function has derivative $\frac{h'(z)}{h(z)}$ by the chain rule, hence $\frac{h'(z)}{h(z)}$ has antiderivative $\mathrm{Log} h(z)$ on γ. Since γ is a closed curve, we thus see from the fundamental theorem of calculus that $\int_\gamma \frac{h'(z)}{h(z)}\, dz = 0$. But recall that $h = f/g$, and hence $\frac{h'(z)}{h(z)} = \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}$ (see earlier exercise concerning the logarithmic derivative). Thus
$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = \int_\gamma \frac{g'(z)}{g(z)}\, dz$$
which implies by the argument principle that $f(\gamma)$ and $g(\gamma)$ do have the same winding number around the origin, as desired. $\square$

- **Example.** The function $f(z) := z^5$ has a quintuple zero at the origin. Now consider the function $g(z) := z^5 + z + 1$. If we look on the curve $\gamma(t) = 2e^{it} : 0 \leq t \leq 2\pi$, which traverses the circle $\{z \in \mathbf{C} : |z| = 2\}$, we see that on this curve
$$|f(z)| = |z|^5 = 2^5 = 32$$
whereas
$$|g(z) - f(z)| = |z+1| \leq |z| + 1 = 2 + 1 = 3$$

(here we have used the triangle inequality $|z + w| \leq |z| + |w|$). Thus the conditions of Rouche's theorem are satisfied, and the total number of zeroes minus poles of g inside γ must match that of f, which is equal to 5. Since g is a polynomial, it certainly doesn't have any poles, and hence g has five zeroes inside γ. To put it another way, on the curve γ, the term z^5 in g is so much larger than the other two that it dominates the function g, and hence by Rouche's theorem g has the same number of zeroes as z^5 inside this curve, i.e. it has five zeroes.

- Indeed we can repeat this analysis for larger circles, i.e. $\{z \in \mathbf{C} : |z| = R\}$ for any $R \geq 2$, and conclude that any such circle contains exactly five zeroes of g. Thus g has exactly five zeroes in the complex plane, and they all live in the disk $\{z \in \mathbf{C} : |z| < 2\}$.

- Let's analyze these zeroes a little more. We keep the same function $g(z) := z^5 + z + 1$, but now take the reference function f to be the constant function $f(z) := 1$, and use the curve $\gamma'(t) = \frac{1}{2}e^{it} : 0 \leq t \leq 2\pi$, which traverses the circle $\{z \in \mathbf{C} : |z| = \frac{1}{2}\}$. On this curve we have

  $$|f(z)| = 1$$

  while

  $$|g(z) - f(z)| = |z^5 + z| \leq |z|^5 + |z| = \frac{1}{2^5} + \frac{1}{2} = \frac{17}{32}.$$

  Thus the conditions of Rouche's theorem are again satisfied, and we see that g has the same number of zeroes inside γ' as f does, which is 0 (as 1 is never equal to 0). Thus g has no zeroes in the disk $\{z : |z| < 1/2\}$, which when combined with our previous analysis shows that all five zeroes must lie within the annulus $\{z : 1/2 < |z| < 2\}$. (Note that g cannot have a zero on either γ or γ' by the argument contained in the proof of Rouche's theorem). In contrast with the behavior on large circles, what is happening on small circles such as $\{z : |z| = 1/2\}$ is that the constant term 1 of $z^5 + z + 1$ is dominating, and so g has the same number of zeroes as 1 (as opposed to z^5, which is what happens for very large circles).

- One might hope to continue this analysis further and pinpoint exactly where the zeroes of f lie. Unfortunately for intermediate circles (e.g.

$\{z : |z| = 1\}$) no single term in g is dominant, and it is not obvious how to use Rouche's theorem to count how many zeroes lie inside the unit circle. Like many tools in complex analysis, Rouche's theorem is not a magic wand, but can give a lot of useful information nevertheless.

- One consequence of Rouche's theorem is that it gives yet another proof of

- **Fundamental theorem of algebra.** Every polynomial $P(z)$ of degree n has exactly n zeroes (counting multiplicity), and thus can be factored completely into linear factors.

- **Proof.** Write the polynomial as

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$$

and note that a_n is non-zero (otherwise P would have degree less than n). Now choose a large circle $\{z : |z| = R\}$ for some $R \gg 1$. On this circle we compare P against the function $f(z) := a_n z^n$. Note that

$$|f(z)| = |a_n z^n| = |a_n| R^n$$

and

$$|P(z) - f(z)| = |a_{n-1} z^{n-1} + \ldots + a_0| \leq |a_{n-1}| R^{n-1} + \ldots + |a_0|.$$

Thus

$$\frac{|P(z) - f(z)|}{|f(z)|} \leq \frac{|a_{n-1}|}{|a_n| R} + \frac{|a_{n-2}|}{|a_n| R^2} + \ldots + \frac{|a_0|}{|a_n| R^n}$$

(recall that $|a_n|$ is non-zero). As $R \to +\infty$, everything on the right-hand side tends to zero. Thus there exists some R_0 such that for every $R > R_0$, we have

$$\frac{|a_{n-1}|}{|a_n| R} + \frac{|a_{n-2}|}{|a_n| R^2} + \ldots + \frac{|a_0|}{|a_n| R^n} < 1$$

and hence

$$|P(z) - f(z)| < |f(z)|.$$

Thus by Rouche's theorem, P and f have the same number of zeroes in the circle $\{z \in \mathbf{C} : |z| = R\}$. But f clearly has n zeroes in this circle, and thus P must also. Letting R go to infinity we obtain the result.$\square$.

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Stability of ODE

- We now give an application of the above machinery, to analyzing first-order linear ordinary differential equations (ODE). Specifically, we consider a system of n (real or complex-valued) functions $x_1(t), \ldots, x_n(t)$, depending on a time parameter t, which evolves by the system of equations

$$\begin{aligned} \frac{dx_1}{dt}(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \ldots + a_{1n}x_n(t) \\ \frac{dx_2}{dt}(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \ldots + a_{2n}x_n(t) \\ &\vdots \\ \frac{dx_n}{dt}(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \ldots + a_{nn}x_n(t) \end{aligned}$$

where $a_{11}, a_{12}, \ldots, a_{nn}$ are fixed (real or complex) numbers, called the *co-efficients* of the ODE. A simple example of such an ODE is the scalar case $n = 1$, which is just the equation

$$\frac{dx_1}{dt}(t) = a_{11}x_1(t)$$

which is the exponential growth equation (if a_{11} is positive), or the exponential decay equation (if a_{11} is negative), or the constant equation (if a_{11} is zero). An example of a more complex system is

$$\begin{aligned} \frac{dx_1}{dt} &= 2x_1 - x_2 \\ \frac{dx_2}{dt} &= 2x_2 - x_3 \\ \frac{dx_3}{dt} &= 2x_3 - x_1; \end{aligned}$$

thus x_1 has both an exponential growth factor (the $2x_1$ term on the right-hand side) and a countervailing decay factor $-x_2$, and similarly for x_2 and x_3. Now it is not clear whether this system will lead to exponential growth or exponential decay. Other situations are possible; for instance, the system

$$\begin{aligned} \frac{dx_1}{dt} &= -x_2 \\ \frac{dx_2}{dt} &= x_1 \end{aligned}$$

has the oscillating solution $x_1(t) = \cos(t)$, $x_2(t) = \sin(t)$, which neither grows nor decays exponentially but instead oscillates forever.

- These types of ODE arise naturally in the study of *feedback systems* in engineering or biology - for instance, there might be n species, and $x_j(t)$ represents the population of the j^{th} species at time t. The various coefficients a_{ij} could be positive or negative and measure the relationship between species i and j (symbiotic, predatory, competitive, etc.) Or the $x_j(t)$ could represent the oscillation of the j^{th} joint on (say) a bridge, etc. with each type of oscillation capable of reinforcing or canceling another. (These are of course simplified models because they can only deal with linear interactions and not non-linear ones, but they are still a very useful first approximation in real-life problems). They also arise in solving higher-order linear ODE such as

$$f'''(t) + 5f''(t) - 3f'(t) + 6f(t) = 0$$

since after the substitution

$$x_1(t) = f(t); \quad x_2(t) = f'(t); \quad x_3(t) = f''(t)$$

we can rewrite the above third-order ODE as a first order system

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= x_3 \\ \frac{dx_3}{dt} &= -5x_3 + 3x_2 - 6x_1 \end{aligned}$$

(why is this system equivalent to the previous ODE after substitution?).

- There are many interesting questions associated with ODE (or with partial differential equations (PDE), for that matter), but we shall be concerned with just one: given a system of ODE of the above form, is it possible for the solution to grow exponentially as $t \to +\infty$ (as it does with e.g. the ODE $\frac{dx}{dt} = 5x$)? If this occurs, we say that the ODE is *exponentially unstable*, which is bad news if you are designing a bridge or managing an ecosystem. There are other possibilities: polynomial instability (the system grows like a polynomial, as e.g. the system $\frac{dx_1}{dt} = x_2; \frac{dx_2}{dt} = 0$ will do), stability (the system oscillates, as $\frac{dx_1}{dt} = -x_2; \frac{dx_2}{dt} = x_1$ will do), and exponential stability (the solution decays exponentially, as e.g. $\frac{dx}{dt} = -5x$ will do). A full discussion of stability analysis is beyond the scope of this course. However, we shall see that the argument principle (and some linear algebra) can be used to address this question of stability.

- To begin with, we use matrix notation to write the system of ODE in a more compact form. If we let $\vec{x}(t)$ denote the vector

$$\vec{x}(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

and let A denote the constant (i.e. time-independent) $n \times n$ matrix

$$A := \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

then the ODE can be written in matrix form as

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t).$$

Now suppose A has an eigenvector v with eigenvalue $\lambda \in \mathbf{C}$, thus $Av = \lambda v$. Then we see that the function

$$\vec{x}(t) := e^{\lambda t} v$$

will be a particular solution to the ODE. Indeed we have

$$\frac{d}{dt}\vec{x}(t) = \frac{d}{dt}e^{\lambda t}v = e^{\lambda t}\lambda v = e^{\lambda t}Av = A\vec{x}(t).$$

For instance, if we write the ODE $\frac{dx_1}{dt} = -x_2$; $\frac{dx_2}{dt} = x_1$ in matrix form as

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

and observe that we have an eigenvalue at $+i$:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = i\begin{pmatrix} 1 \\ i \end{pmatrix}$$

then we see that

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

will be a solution to the ODE. Note that as $t \to +\infty$, e^{it} oscillates but stays bounded, and so the same will happen with $x_1(t)$ and $x_2(t)$. This is because the eigenvalue $\lambda = i$ that we chose here was purely imaginary.

- More generally, any purely imaginary eigenvalue will lead to an oscillating solution. But an eigenvalue with positive real part, e.g. $\lambda = x + iy$ with $x > 0$, will lead to an exponentially growing solution, since $e^{\lambda t} = e^{xt}e^{iyt}$. Similarly, an eigenvalue with negative real part will lead to an exponentially decaying solution. This turns out to be the more or less the complete answer to when an ODE is stable or not:

- **Stability Criterion.** If at least one of the eigenvalues of A has a positive real part, then the ODE $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ has exponentially growing solutions. If however all the eigenvalues of A have negative real part, then all the solutions of $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ decay exponentially. (If there are purely imaginary eigenvalues, the situation is more complicated - one may have stability or polynomial instability, depending on something called the Jordan canonical form of A, which is beyond the scope of this course).

- We will not prove this criterion here, but it will be covered in any good course on differential equations or advanced linear algebra. If we accept this criterion as valid, then it basically reduces the question of whether an ODE is (exponentially) stable or unstable to the question of whether the eigenvalues of A all lie on the negative half-plane $\{z \in \mathbf{C} : \mathrm{Re} z < 0\}$, or whether there is an eigenvalue in the positive half-plane $\{z \in \mathbf{C} : \mathrm{Re} z > 0\}$ (ignoring for now the issue of eigenvalues on the imaginary axis).

- Of course, this begs the question of how one finds eigenvalues. From basic linear algebra we know that eigevalues λ are nothing more than the zeroes to the *characteristic polynomial*

$$f(z) := \det(A - zI)$$

which is a polynomial of degree n and thus has n zeroes (counting multiplicity) by the fundamental theorem of algebra. The polynomial f is quite easy to compute if one knows A, but finding its zeroes can be very difficult (for instance, if $n \geq 5$, it has been proven that there is no general formula for the zeroes in terms of the arithmetic operations and taking k^{th} roots). However, by the argument principle we can count how many zeroes there are in the negative half-plane; if all n of the zeroes are there then we have stability, and if at least one is in the positive half-plane then we have instability. This is best explained by an example.

- **Example.** Suppose that

$$A := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

Then the characteristic polynomial is

$$f(z) := \det \begin{pmatrix} 1-z & 1 & 0 \\ 0 & -z & 1 \\ 1 & -1 & -z \end{pmatrix} = (1-z)(z^2+1)+1\times 1 = -z^3+z^2-z+2.$$

Now we find how many zeroes f has in the left half-plane. We begin by choosing a large radius R and considering the C-shaped contour that goes up from $-Ri$ to Ri via the straight line $\gamma_1(t) = it : -R \leq t \leq R$, and then around the semi-circle $\gamma_2(t) = Re^{it} : \pi/2 \leq t \leq 3\pi/2$. The combined contour $\gamma := \gamma_1 + \gamma_2$ is a simple closed curve traversed once anti-clockwise, and as $R \to +\infty$ it contains more and more of the negative half-plane.

- By the argument principle, the number of zeroes of f inside γ is equal to the number of times $f(\gamma) = f(\gamma_1) + f(\gamma_2)$ winds around the origin. So we shall plot $f(\gamma_1)$ and $f(\gamma_2)$.

- Since $\gamma_1(t) = it$ for $-R \leq t \leq R$, we see that $f(\gamma_1(t)) = -(it)^3 + (it)^2 - it + 2 = i(t^3 - t) + (2 - t^2)$. Now we draw a sign diagram, and observe that $t^3 - t$ is positive for $t > 1$ and $-1 < t < 0$ and negative for $t < -1$ and $0 < t < 1$, while $2 - t^2$ is positive for $-\sqrt{2} < t < \sqrt{2}$

and negative for $t < -\sqrt{2}$ and $t > \sqrt{2}$. Also, as $t \to \pm\infty$ the angle of the curve becomes more and more vertical (because t^3 becomes larger in magnitude than all the other terms put together). This allows one to sketch $f(\gamma_1)$ (it looks like a vertical line bent and twisted to create a loop near the origin). Now look at $f(\gamma_2)$, which is parameterized by

$$f(\gamma_2(t)) = -(Re^{it})^3 + (Re^{it})^2 - (Re^{it}) + 2 = -R^3 e^{3it} + \text{ terms of size } R^2 \text{ or less.}$$

When R gets large then R^3 becomes much larger than R^2 and we can effectively ignore the error terms. The curve $-R^3 e^{3it} : \pi/2 \leq t \leq 3\pi/2$ starts at $R^3 i$ and winds one and a half times anti-clockwise around the origin until it reaches $-R^3 i$. Gluing this with the sketch of $f(\gamma_1)$ we see that $f(\gamma)$ winds twice around the origin, and never actually passes through the origin. Thus f has two zeroes on the negative half-plane and no zeroes on the imaginary axis, which means that there must be at least one zero on the right half-plane and hence the ODE is exponentially unstable.