Math 3228, Assignment 3 solutions.

- Q1. Let $P(z) := a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$ be a polynomial of degree n such that all the coefficients $a_0, \ldots, a_n$ are real. Show that if z is a zero of P, then $\overline{z}$ is also a zero of P. (Thus zeroes either live on the real line, or come in pairs, one above the real line and one below the real line).

- A1. Suppose that z is a zero of P, then

$$a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 = 0.$$

We can take conjugates (using the facts $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{z^n} = \overline{z}^n$ to obtain that

$$\overline{a_n} \overline{z}^n + \overline{a_{n-1}} \overline{z}^{n-1} + \ldots + \overline{a_0} = 0.$$

But since $a_0, \ldots, a_n$ are all real, we thus have

$$a_n \overline{z}^n + a_{n-1} \overline{z}^{n-1} + \ldots + a_0 = 0,$$

and so $\overline{z}$ is a zero of P.

- Q2. Let $P(z)$ be the polynomial $P(z) = z^3 + z^2 + 4z + 30$. Determine how many zeroes P has on each of the four quadrants, and on each of the four co-ordinate axes.

- A2. Rather unintentionally, this polynomial has the explicit factorization

$$P(z) = (z+3)(z-1+3i)(z-1-3i)$$

which makes the question rather easy; however the method I describe below works even when there is no obvious factorization for the polynomial. Let us begin by working out how many zeroes there are on the imaginary axis. Since

$$P(iy) = -iy^3 - y^2 + 4iy + 30$$

we see that $P(iy)$ crosses the real axis when $-y^3 + 4y = 0$, i.e. when $y = -2, 0, +2$, while $P(iy)$ crosses the imaginary axis when $-y^2 + 30 = 0$, i.e. when $y = \pm\sqrt{30}$. Thus for no value of y is the real and imaginary parts of P simultaneously zero, so there are no zeroes on the imaginary axis. Now consider what $P(iy)$ does from $y = -R$ to $y = +R$ for a large $R > 0$; from the above discussion it starts off in the second quadrant (near iR^3), crosses into the first quadrant, then makes a small loop from the first to the fourth back to the first and then back to the fourth again, and then finally crosses into the third quadrant ending up near $-iR^3$. In particular the total change in argument is approximately $-\pi$. If one then executes an anti-clockwise circle from iR back to $-iR$, i.e. $z = Re^{i\theta}$, $\pi/2 \leq \theta \leq 3\pi/2$, then $P(z)$ is approximately $R^3 e^{3\pi i\theta}$ (plus smaller errors of size R^2 or so), for a total change in argument of 3π. Thus in total this contour has an argument change of 2π, i.e. it goes once around the origin. By the argument principle (letting $R \to \infty$) this means that there is only one zero on the left half of the complex plane, which must therefore be on the negative real axis by Q1. On the right half plane this leaves two remaining zeroes (the fundamental theorem of algebra ensures P has three zeroes), but they can't lie on the positive real axis because P is positive there. Thus by Q1 again P must have one zero in both the first and fourth quadrants.

- Q3. Find a conformal mapping that takes the half-disk $\{z \in \mathbf{C} : |z| < 1; \mathrm{Im}(z) > 0\}$ to the half-plane $\{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$.

- A3. The map $f(z) = (z+1)^2/(z-1)^2$, or more generally $f(z) = C(z+1)^2/(z-1)^2$ for any positive $C > 0$ will work; as will the map $f(z) = -(z-1)^2/(z+1)^2$, or more generally $f(z) = -C(z+1)^2/(z-1)^2$. This answer can be obtained by first converting the half-disk into one of the four co-ordinate quadrants and then squaring that quadrant. These maps are analytic and their derivatives are non-zero on the half-disk and so they are conformal. Note that if you square the half-disk first, hoping to convert into a whole disk, you will just barely fail because the half-disk is open (it doesn't contain the real axis) and so its square does not intersect the positive real axis.

- Q4. If $f(z) = f(x+iy)$ is a differentiable function of x and y separately (but not necessarily a differentiable function of z, define the derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$ by
$$\frac{\partial f}{\partial z} := \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}); \quad \frac{\partial f}{\partial \overline{z}} := \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}).$$

- (a) Show that if f is complex analytic on C, that $\frac{\partial f}{\partial \overline{z}} = 0$ and $\frac{\partial f}{\partial z}(z) = f'(z)$.

- A4(a). This can be proven by direct algebraic computation using the Cauchy-Riemann equations; but let me indicate another proof. The starting point is the Newton approximation

$$f(x + dx + i(y + dy)) \approx f(x + iy) + dx\frac{\partial f}{\partial x}(x + iy) + dy\frac{\partial f}{\partial y}(x + iy);$$

  more precisely we have

$$\lim_{dx,dy \to 0} \frac{|f(x + dx + i(y + dy)) - f(x + iy) - dx\frac{\partial f}{\partial x}(x + iy) - dy\frac{\partial f}{\partial y}(x + iy)}{|x + iy|} = 0$$

  (strictly speaking this requires f to be continuously differentiable, not just differentiable, though analytic functions are automatically continuously differentiable to any order). Writing $dz := dx + idy$ we thus see from the definitions that

$$f(z + dz) \approx f(z) + dz\frac{\partial f}{\partial z}(z) + \overline{dz}\frac{\partial f}{\partial \overline{z}}(z);$$

  this gives an intuitive explanation as to where $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$ come from. On the other hand, if f is analytic then

$$f(z + dz) \approx f(z) + dzf'(z),$$

  and comparing the two gives (a). (This is not a rigorous proof yet, but can be made rigorous by using limits as indicated above).

- (b) Show that if $f : \mathbf{C} \to \mathbf{C}$ and $g : \mathbf{C} \to \mathbf{C}$ are differentiable functions, then the derivatives of the composition $f \circ g : \mathbf{C} \to \mathbf{C}$ are given by

$$\frac{\partial f \circ g}{\partial z}(z) = \frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial z}(z) + \frac{\partial f}{\partial \overline{z}}(g(z))\overline{\frac{\partial g}{\partial \overline{z}}(z)}$$

  and

$$\frac{\partial f \circ g}{\partial \overline{z}}(z) = \frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial \overline{z}}(z) + \frac{\partial f}{\partial \overline{z}}(g(z))\overline{\frac{\partial g}{\partial z}(z)}.$$

- A4(b). One has to take care here because g has both real and imaginary parts, and $f \circ g$ depends on both. If we write $g = u + iv$ then the chain rule gives

$$\frac{\partial f \circ g}{\partial x}(z) = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x},$$

and thus

$$\frac{\partial f \circ g}{\partial x}(z) = \frac{\partial f}{\partial x}(g(z))\frac{\partial \mathrm{Re} g}{\partial x}(z) + \frac{\partial f}{\partial y}(g(z))\frac{\partial \mathrm{Im}(g)}{\partial x}(z).$$

There is a similar formula for the y derivative, which will eventually give the above formulae. Another way to proceed is by Newton approximation. Starting with

$$g(z + dz) \approx g(z) + dz\frac{\partial g}{\partial z}(z) + \overline{dz}\frac{\partial g}{\partial \overline{z}}(z);$$

we see that

$$dg = dz\frac{\partial g}{\partial z}(z) + \overline{dz}\frac{\partial g}{\partial \overline{z}}(z)$$

and taking conjugates

$$\overline{dg} = \overline{dz}\,\overline{\frac{\partial g}{\partial z}(z)} + dz\,\overline{\frac{\partial g}{\partial \overline{z}}(z)}.$$

On the other hand, from Newton's approximation again we have

$$f(g(z) + dg) \approx f(g(z)) + dg\frac{\partial f}{\partial z}(g(z)) + \overline{dg}\frac{\partial f}{\partial \overline{z}}(g(z)),$$

and combining this with the previous estimates we see that

$$df \approx dz\left(\frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial z}(z) + \frac{\partial f}{\partial \overline{z}}(g(z))\overline{\frac{\partial g}{\partial \overline{z}}(z)}\right)$$

$$+\overline{dz}\left(\frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial \overline{z}}(z) + \frac{\partial f}{\partial \overline{z}}(g(z))\overline{\frac{\partial g}{\partial z}(z)}\right).$$

Comparing this with the Newton approximation

$$f \circ g(z + dz) \approx f \circ g(z) + dz\frac{\partial f \circ g}{\partial z}(z) + \overline{dz}\frac{\partial f \circ g}{\partial \overline{z}}(z)$$

we obtain the result.

- (c) A twice-differentiable function f is called *harmonic* on $\mathbf{C}$ if $\frac{\partial^2 f}{\partial x^2}(z) + \frac{\partial^2 f}{\partial y^2}(z) = 0$ for all $z \in \mathbf{C}$. Show that f is harmonic if and only if $\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} f = 0$. Conclude in particular that every complex analytic function is harmonic.

- A4(c). I apologize here; the statement is true as stated, but is far more difficult to prove unless one already assumes that f is twice *continuously* differentiable, since this implies that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ (Clairaut's theorem). Assuming this it is easy to see that

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} f = \frac{1}{4}(\frac{\partial^2 f}{\partial x^2}(z) + \frac{\partial^2 f}{\partial y^2}(z))$$

  and the claim follows. Note that if f is analytic then $\frac{\partial}{\partial \overline{z}} f$ is zero by 4(a) and hence $\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} f = 0$.

- (d) Suppose that $f : \mathbf{C} \to \mathbf{C}$ is complex analytic, and $g : \mathbf{C} \to \mathbf{C}$ is harmonic. Show that $g \circ f : \mathbf{C} \to \mathbf{C}$ also harmonic.

- A4(d). By Q4(c), it will suffice to show that

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}(g \circ f) = 0.$$

  First observe from Q4(a) and Q4(b)

$$\frac{\partial}{\partial \overline{z}}(g \circ f) = \frac{\partial g}{\partial \overline{z}}(f(z))\overline{f'(z)},$$

  so by the product rule

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}(g \circ f) = \frac{\partial}{\partial z}(\frac{\partial g}{\partial \overline{z}}(f(z)))\overline{f'(z)}$$

$$+\frac{\partial g}{\partial \overline{z}}(f(z))\frac{\partial}{\partial z}\overline{f'(z)}.$$

  Observe that f' is analytic, hence by conjugating Q4(a) that

$$\frac{\partial}{\partial z}\overline{f'(z)} = 0,$$

so the second term vanishes. For the first term, we use Q4(b) and Q4(a) to expand

$$\frac{\partial}{\partial z}(\frac{\partial g}{\partial \overline{z}}(f(z))) = \frac{\partial^2 g}{\partial z \partial \overline{z}}(f(z))f'(z) = 0$$

since g is harmonic.

- Q5. Determine all the residues at each of the poles $0, -1, -2, \ldots$ of the Gamma function.

- A5. Since $\Gamma(z) = \Gamma(z+1)/z$, and $\Gamma(1) = 1$, we see that $\Gamma(z)$ has a Laurent expansion of the form $\Gamma(z) = (1 + a_1 z + a_2 z^2 + \ldots)/z$ around 0 and hence has a simple pole at 0 with residue 1. Thus $\Gamma(z+1) = 1/(z+1) + \ldots$ has a simple pole at -1 with residue 1. Since $1/z$ is analytic and equals -1 at -1, we see that $\Gamma(z) = \Gamma(z+1)/z$ has a simple pole at -1 with residue -1. Thus $\Gamma(z+1)$ has a simple pole at -2 with residue -1, which dividing by z again shows that $\Gamma(z)$ has a simple pole at -2 with residue $1/2$. Continuing this way by induction we see that $\Gamma(z)$ has a simple pole at $-n$ with residue $(-1)^n/n!$.

- Q6. Let C be the circle $C := \{x + iy : (x-a)^2 + (y-b)^2 = r^2\}$ centered at $a + bi$ with radius r, where we assume that $r > 0$ and $r \neq \sqrt{a^2 + b^2}$. Show (either by algebraic means, or geometric means) that the image of C under the inversion map $z \mapsto \frac{1}{z}$ is still a circle, and determine its center and radius.

- A6. It is easiest to work using complex co-ordinates $z := x + iy$ and $z_0 := a + ib$, so the equation for the circle becomes $|z - z_0|^2 = r^2$. Now we map $w := 1/z$, thus $z = 1/w$ and the equation for w is

  $$|1/w - z_0|^2 = r^2.$$

  Multiplying by $|w|^2$, we obtain

  $$|1 - z_0 w|^2 = r^2 |w|^2;$$

  using the identity $|w|^2 = w\overline{w}$ this becomes

  $$1 - z_0 w - \overline{z_0 w} + |z_0|^2 |w|^2 = r^2 |w|^2,$$

completing the square we get

$$|w|^2 - \frac{z_0}{|z_0|^2 - r^2} w - \frac{\overline{z_0}}{|z_0|^2 - r^2} \overline{w} + \frac{|z_0|^2}{(|z_0|^2 - r^2)^2} = \frac{r^2}{(|z_0|^2 - r^2)^2}.$$

The left hand side is $|w - \frac{z_0}{|z_0|^2 - r^2}|^2$, so this is the equation for a circle with center $z_0/(|z_0|^2 - r^2)$ and radius $r/||z_0|^2 - r^2|$. Note that one can easily reverse these steps and ensure that every point on this image circle did indeed come from a point on the original circle.

- Q7. Let z, w be any complex numbers with $\mathrm{Re}(z), \mathrm{Re}(w) > 0$. Show that
$$\int_0^1 t^{z-1}(1-t)^{w-1}\, dt = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

- A7. A modification of page 15 of Week 8 notes gives
$$\Gamma(z)\Gamma(w) = (\int_0^1 y^{z-1}(1-y)^{w-1}\, dy)(\int_0^\infty u^{z+w-1} e^{-u}\, du)$$
and the claim follows.

- Q8. Use the residue theorem applied to the function $f(z) := \frac{1}{z^2 \tan \pi z}$ and the contour which is a square connecting $R - Ri$, $R + Ri$, $-R + Ri$, and $-R - Ri$ where $R = m + \frac{1}{2}$ is a large half-integer, to deduce that
$$\frac{\pi^2}{6} = \zeta(2) = \sum_{n=1}^\infty \frac{1}{n^2}.$$

- A8. The function f has poles at the integers $\mathbf{Z}$ (since this is where z and $\tan \pi z$ can vanish). Let us first work out the orders and residues of these poles. We begin with the pole at 0. Since $\tan \pi z = \sin \pi z / \cos \pi z$ has a simple zero at zero, f has a triple pole at zero. To compute the residue, we begin with the Taylor expansions
$$\sin \pi z = \pi z - \pi^3 z^3/3! + \ldots$$
and
$$\cos \pi z = 1 - \pi^2 z^2/2! + \ldots,$$

which on dividing gives

$$\cos \pi z / \sin \pi z = \frac{1}{\pi z} - \frac{\pi}{3} z + \ldots$$

and hence

$$f(z) = \frac{1}{\pi z^3} - \frac{\pi}{3z} + \ldots,$$

i.e. the residue of f at 0 is $-\pi/3$.

Now let us consider the residue at any other integer n. Again we Taylor expand, but this time around n:

$$\sin \pi z = (-1)^n \pi (z - n) + \ldots$$

$$\cos \pi z = (-1)^n + \ldots$$

$$z^2 = n^2 + \ldots$$

and hence

$$f(z) = \frac{1}{n^2 \pi (z - n)} + \ldots$$

and hence the residue at 0 is $\frac{1}{n^2 \pi}$. Thus by the residue theorem, the integral on the contour γ_m indicated is

$$\int_{\gamma_m} f(z)\ dz = 2\pi i(-\pi/3 + \sum_{n=1}^{m} \frac{1}{n^2 \pi} + \sum_{n=-m}^{-1} \frac{1}{n^2 \pi})$$

which simplifies to

$$4i(\sum_{n=1}^{m} \frac{1}{n^2} - \frac{\pi^2}{6}).$$

So it will suffice to show that $\int_{\gamma_m} f(z)\ dz$ goes to zero as $m \to \infty$.

- Let us see what f does on, say, the top edge of the contour, where $z = (m + \frac{1}{2})i + x$ for some $-m - \frac{1}{2} \leq x \leq m + \frac{1}{2}$. Write

$$\left|\frac{1}{\tan(\pi z)}\right| = \left|\frac{\cos(\pi z)}{\sin(\pi z)}\right| = \left|\frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}}\right|.$$

Substituting $z = (m + \frac{1}{2})i + x$, we obtain

$$|\frac{1}{\tan(\pi z)}| = |\frac{e^{-\pi(m+1/2)}e^{\pi ix} + e^{\pi(m+1/2)}e^{-\pi ix}}{e^{-\pi(m+1/2)}e^{\pi ix} - e^{\pi(m+1/2)}e^{-\pi ix}}|$$

which simplifies to

$$|\frac{1}{\tan(\pi z)}| = |\frac{1 + e^{-\pi(2m+1)}e^{2\pi ix}}{1 - e^{-\pi(2m+1)}e^{2\pi ix}}| \le \frac{1 + e^{-\pi(2m+1)}}{1 - e^{-\pi(2m+1)}}.$$

Since $m \ge 1$, we have $\frac{1+e^{-\pi(2m+1)}}{1-e^{-\pi(2m+1)}} \le 2$ (say). Thus $\frac{1}{\tan \pi z}$ is bounded in magnitude by 2 on the top edge of the contour. The bottom edge is similar (in fact, it follows from the top edge bound because tan is an odd function), so now consider the right edge where $z = (m + \frac{1}{2}) + iy$. Now we have

$$|\frac{1}{\tan(\pi z)}| = |\frac{ie^{-\pi y} - ie^{\pi y}}{ie^{-\pi y} + ie^{\pi y}}| = \frac{1 - e^{-2\pi y}}{1 + e^{2\pi y}} \le 1.$$

Similarly for the left side. Thus in all cases we have $|\frac{1}{\tan \pi z}| \le 2$ on the contour γ_m, and hence $|\frac{1}{z^2 \tan \pi z}| \le \frac{2}{R^2}$. Since the contour has total length $8R$, the integral is thus at most $\frac{16}{R}$, which does indeed go to zero as $R \to +\infty$, as desired.