Math 3228, Assignment 3 (Due September 22, either in my office JD2134B or in class). This assignment counts for 25% of the final mark for the subject.

- Q1. Let $P(z) := a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$ be a polynomial of degree n such that all the coefficients $a_0, \ldots, a_n$ are real. Show that if z is a zero of P, then $\overline{z}$ is also a zero of P. (Thus zeroes either live on the real line, or come in pairs, one above the real line and one below the real line).

- Q2. Let $P(z)$ be the polynomial $P(z) = z^3 + z^2 + 4z + 30$. Determine how many zeroes P has on each of the four quadrants, and on each of the four co-ordinate axes. (Hint: first work out how many zeroes P has on the imaginary axis, on the left half-plane, and on the right-half plane. Then use Q1. Also observe that $P(z)$ is positive when z is on the positive real axis.)

- Q3. Find a conformal mapping that takes the half-disk $\{z \in \mathbf{C} : |z| < 1; \mathrm{Im}(z) > 0\}$ to the half-plane $\{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$. (Hint: Möbius transformations will not be enough to achieve this task by themselves, but you can find a Möbius transform to map the half-disk to a quadrant, then use the squaring map $z \mapsto z^2$).

- Q4. If $f(z) = f(x+iy)$ is a differentiable function of x and y separately (but not necessarily a differentiable function of z, define the derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$ by

$$\frac{\partial f}{\partial z} := \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}); \quad \frac{\partial f}{\partial \overline{z}} := \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}).$$

- (a) Show that if f is complex analytic on C, that $\frac{\partial f}{\partial \overline{z}} = 0$ and $\frac{\partial f}{\partial z}(z) = f'(z)$. (Hint: use the Cauchy-Riemann equations).

- (b) Show that if $f : \mathbf{C} \to \mathbf{C}$ and $g : \mathbf{C} \to \mathbf{C}$ are differentiable functions, then the derivatives of the composition $f \circ g : \mathbf{C} \to \mathbf{C}$ are given by

$$\frac{\partial f \circ g}{\partial z}(z) = \frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial z}(z) + \frac{\partial f}{\partial \overline{z}}(g(z))\overline{\frac{\partial g}{\partial \overline{z}}(z)}$$

and

$$\frac{\partial f \circ g}{\partial \overline{z}}(z) = \frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial \overline{z}}(z) + \frac{\partial f}{\partial \overline{z}}(g(z))\overline{\frac{\partial g}{\partial z}(z)}.$$

(Hint: use the ordinary chain rule (in several variables), and rewrite x and y derivatives in terms of z and $\overline{z}$ derivatives).

- (c) A twice-differentiable function f is called *harmonic* on $\mathbf{C}$ if $\frac{\partial^2 f}{\partial x^2}(z) + \frac{\partial^2 f}{\partial y^2}(z) = 0$ for all $z \in \mathbf{C}$. Show that f is harmonic if and only if $\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} f = 0$. Conclude in particular that every complex analytic function is harmonic.

- (d) Suppose that $f : \mathbf{C} \to \mathbf{C}$ is complex analytic, and $g : \mathbf{C} \to \mathbf{C}$ is harmonic. Show that $f \circ g : \mathbf{C} \to \mathbf{C}$ also harmonic. (Hint: use (a), (b), (c). You will need the fact that if f is complex analytic then its derivative f' is also complex analytic).

- Q5. Determine all the residues at each of the poles $0, -1, -2, \ldots$ of the Gamma function.

- Q6. Let C be the circle $C := \{x + iy : (x-a)^2 + (y-b)^2 = r^2\}$ centered at $a + bi$ with radius r, where we assume that $r > 0$ and $r \neq \sqrt{a^2 + b^2}$. Show (either by algebraic means, or geometric means) that the image of C under the inversion map $z \mapsto \frac{1}{z}$ is still a circle, and determine its center and radius.

- Q7. Let z, w be any complex numbers with $\mathrm{Re}(z), \mathrm{Re}(w) > 0$. Show that
$$\int_0^1 t^{z-1}(1-t)^{w-1}\ dt = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$
Hint: this is a modification of one of the arguments in the notes.

- Q8. Use the residue theorem applied to the function $f(z) := \frac{1}{z^2 \tan \pi z}$ and the contour which is a square connecting $R - Ri$, $R + Ri$, $-R + Ri$, and $-R - Ri$ where $R = m + \frac{1}{2}$ is a large half-integer, to deduce that
$$\frac{\pi^2}{6} = \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$
(Hint: f has simple poles at every integer, except at 0 where it has a triple pole. To show that the integral on the contour goes to zero, first prove that $|\frac{1}{\tan \pi z}| < 2$ for all z on the square.)