Mathematics 133  
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Instructions: Try to do all five problems; they are all of equal value. There is plenty of working space, and a blank page at the end.  

You may enter in a nickname if you want your midterm score posted.  

Good luck!  

Name: ____________________________  

Nickame: __________________________  

Student ID: _________________________  

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Problem 1. ____________  

Problem 2. ____________  

Problem 3. ____________  

Problem 4. ____________  

Problem 5. ____________  

Total: _________________________
Problem 1. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be the function defined by setting \( f(x) = 1 \) when \( 0 \leq x \leq \pi \), and \( f(x) = 0 \) when \(-\pi < x < 0\), and then extending periodically by \( 2\pi \) (thus for instance \( f(x) = 1 \) when \( 2\pi \leq x \leq 3\pi \)); such a function is sometimes called a square wave. Compute the Fourier coefficients \( \hat{f}(n) \) of \( f \) (Caution: the case \( n = 0 \) may have to be treated separately. You may find the identity \( e^{\pi in} = (-1)^n \) useful). Using Parseval’s identity, conclude that

\[
\sum_{n \in \mathbb{Z} : n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{4}.
\]

First we compute the zeroth Fourier mode \( \hat{f}(0) \):

\[
\hat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2\pi} \int_{0}^{\pi} 1 \, dx = \frac{1}{2\pi} \pi = \frac{1}{2}.
\]

Next, we compute the non-zero Fourier modes \( \hat{f}(n), n \neq 0 \):

\[
\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx = \frac{1}{2\pi} \int_{0}^{\pi} e^{-inx} \, dx = \frac{1}{2\pi} \left[ \frac{e^{-inx}}{-in} \right]_{0}^{\pi} = \frac{1 - e^{-in\pi}}{2\pi in} = \frac{1 - (-1)^n}{2\pi in}.
\]

In particular we see that \( \hat{f}(n) = 0 \) for even \( n \), and \( \hat{f}(n) = \frac{1}{\pi in} \) for odd \( n \). In other words

\[
f(x) \sim \frac{1}{2} + \sum_{n \in \mathbb{Z} : n \text{ odd}} \frac{1}{\pi in} e^{inx}.
\]

By Parseval’s identity we thus have

\[
\|f\|^2 = \frac{1}{4} + \sum_{n \in \mathbb{Z} : n \text{ odd}} \left| \frac{1}{\pi in} \right|^2 = \frac{1}{4} + \frac{1}{\pi^2} \sum_{n \in \mathbb{Z} : n \text{ odd}} \frac{1}{n^2}.
\]

But we have

\[
\|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{0}^{\pi} 1 \, dx = \frac{1}{2\pi} \pi = \frac{1}{2}
\]

and hence after a little algebra we obtain

\[
\sum_{n \in \mathbb{Z} : n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{4}
\]

as desired.
**Problem 2.** Let \( f \) be a \( 2\pi \)-periodic function which is continuous and differentiable on the interval \(-\pi < x < \pi\), but has jump discontinuities at \( x = -\pi \) and \( x = \pi\). Suppose also that \( f \) and its derivative remain bounded on the interval \(-\pi < x < \pi\); more precisely, suppose that there exists a constant \( M > 0 \) such that \( |f(x)| \leq M \) and \( |f'(x)| \leq M \) for all \(-\pi < x < \pi\).

(a) Establish the bound
\[
|\hat{f}(n)| \leq \frac{2M}{|n|} \quad \text{for all } n \neq 0.
\]

(Hint: integrate by parts).

Let \( f(\pi^-) \) be the left limit of \( f \) at \( \pi \); observe that \( |f(\pi^-)| \leq M \) since \( |f(x)| \leq M \) for all \(-\pi < x < \pi\). Similarly if we let \( f(-\pi^+) \) be the right limit of \( f \) at \(-\pi \) we have \( |f(-\pi^+)| \leq M \). Now we integrate by parts to obtain
\[
|\hat{f}(n)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \right| \\
= \frac{1}{2\pi} \left| f(\pi^-) \frac{e^{-inx}}{-in} \bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x) \frac{e^{-inx}}{-in} \, dx \right| \\
= \frac{1}{2\pi} \left| f(\pi^-) \frac{e^{-inx}}{-in} - f(-\pi^+) \frac{e^{inx}}{-in} - \int_{-\pi}^{\pi} f'(x) \frac{e^{-inx}}{-in} \, dx \right| \\
\leq \frac{1}{2\pi} (M \frac{1}{n} + M \frac{1}{N} + \left| \int_{-\pi}^{\pi} f'(x) \frac{e^{-inx}}{-in} \, dx \right|) \\
\leq \frac{1}{2\pi} \left( \frac{2M}{n} + \int_{-\pi}^{\pi} \frac{M}{n} \, dx \right) \\
\leq \frac{1}{2\pi} \left( \frac{2M}{n} + \frac{2\pi M}{n} \right) \\
\leq \frac{1}{2\pi} \frac{4\pi M}{n} \\
= \frac{2M}{n}
\]

as desired.

(b) Explain briefly why the series \( \sum_{n=-\infty}^{\infty} |\hat{f}(n)| \) cannot be absolutely convergent. (Hint: Do not try to use (a). Instead, you can use the fact that a uniformly convergent series of continuous functions must be continuous (or, what amounts to much the same thing, you can use the Weierstrass \( M \)-test)).
Suppose for contradiction that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|$ is absolutely convergent. Then the Fourier series $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$ converges absolutely and uniformly to $f$ (by the Weierstrass $M$-test). But since each of the terms in this Fourier series are continuous, this means that $f$ is itself continuous, a contradiction.

Note, by the way, that (a) and (b) together give a (very indirect!) proof that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ must be divergent. One can also use Q1 and Q2(b) to obtain a similar result.
Problem 3. Let \( f \) be a Riemann integrable function, let \( N \geq 0 \) be a non-negative integer, and let \( \sigma_N(f) = \frac{S_0(f) + \cdots + S_{N-1}(f)}{N} \) be the \( N^{th} \) Fejér sum (or Cesáro sum), as defined in page 53 of the textbook. Prove that

\[
\sigma_N(f)(x) = \sum_{n=-N}^{N} (1 - \frac{|n|}{N}) \hat{f}(n)e^{inx}
\]

for all \( x \). (For comparison, recall that \( S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx} \).

We have

\[
\sigma_N(f)(x) = \frac{S_0(f)(x) + \cdots + S_{N-1}(f)(x)}{N} = \frac{1}{N} \sum_{k=0}^{N-1} S_k(f)(x) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{n=-k}^{k} \hat{f}(n)e^{inx}.
\]

Observe that the only values of \( n \) which can appear are when \( |n| \leq k \leq N-1 \). Observe that the term \( \hat{f}(n)e^{inx} \) appears in all the inner sums for which \( N-1 \geq k \geq |n| \), and does not appear in any of the others. For each fixed \( n \), the number of \( k \) between \( N-1 \) and \( |n| \) is \( N-|n| \), thus we have

\[
\sigma_N(f)(x) = \frac{1}{N} \sum_{n=|n|}^{N-1} (N-|n|) \hat{f}(n)e^{inx}.
\]

Note we can freely add the \(|n| = N\) terms to this sum since they are zero. The claim now follows by moving the \( \frac{1}{N} \) factor inside.
Problem 4. A function $f$ is called even if one has $f(x) = f(-x)$ for all $x$, and is called odd if one has $f(x) = -f(-x)$ for all $x$. Show that if $f$ and $g$ are Riemann-integrable $2\pi$-periodic functions which are both odd, then $f * g$ is even. (In other words, the convolution of two odd functions is an even function).

Proof A (sketch): First observe that a function $f$ is even if and only if $f(n) = f(-n)$ for all $n$, and is odd if and only if $f(n) = -f(-n)$ for all $n$; these facts can be shown by comparing the Fourier coefficients of $f(x)$ and $f(-x)$. The claim will then follow from the identity $\hat{f} * \hat{g}(n) = \hat{f}(n)\hat{g}(n)$.

Proof B: For any $x$, we compute

$$f * g(-x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)g(-x - y) \, dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} (-f(-y))(-g(x + y)) \, dy.$$ 

Making the change of variables $z = -y$ (and remembering to change the limits of integration) we obtain

$$f * g(-x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (-f(z))(-g(x - z)) \, (-dz).$$

Reversing the limits of integration and cancelling all the signs, we obtain

$$f * g(-x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z)g(x - z) \, dz = f * g(x)$$

and the claim follows.
Problem 5. Let $u, v$ be elements of a complex inner product space $V$. Show that

$$\langle u, v \rangle = \frac{\|u + v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - \|u - iv\|^2}{4}.$$ 

(This identity, sometimes called the complex parallelogram identity, shows that if one is given the magnitudes of all the vectors in a vector space, one can reconstruct the inner product also.)

Observe that

$$\|u + v\|^2 = \langle u + v, u + v \rangle = \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle.$$ 

Replacing $v$ by $-v$ we obtain

$$\|u - v\|^2 = \langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle.$$ 

Subtracting these two equations we obtain

$$\|u + v\|^2 - \|u - v\|^2 = 2\langle u, v \rangle + 2\langle v, u \rangle.$$ 

Replacing $v$ by $iv$ we obtain

$$\|u + iv\|^2 - \|u - iv\|^2 = 2\langle u, iv \rangle + 2\langle iv, u \rangle;$$

pulling the factors of $i$ out and then multiplying this by $i$ we obtain

$$i\|u + iv\|^2 - i\|u - iv\|^2 = 2\langle u, v \rangle - 2\langle v, u \rangle.$$ 

Adding this to a previous equation we obtain

$$\|u + v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - \|u - iv\|^2 = 4\langle u, v \rangle$$

and the claim follows.