Assignment 3 (Due Feb 5). Covers: pages 51-54, 70-81 of text.

The questions marked "Optional" are more challenging, and will not count toward your final grade. They will however strengthen both your technical skills and your conceptual understanding of the material.

- Q1. Do Exercise 15 of Chapter 2 in the textbook.
- Q2 (Optional). Do Problem 2 (in Section 7) of Chapter 2 in the textbook.
- Q3 (a). Suppose f is 2π-periodic and continuously differentiable. Show that for every $\varepsilon > 0$, there exists a trigonometric polynomial P such that
$$|f(x) - P(x)| < \varepsilon$$
and
$$|f'(x) - P'(x)| < \varepsilon$$
for all $-\pi \leq x \leq \pi$; this is a variant of Corollary 5.4 of Chapter 2 in the textbook, but with a stronger hypothesis (f is not just continuous, but is in fact continuously differentiable) and a stronger conclusion (we not only have P close to f, but we also have P' close to f'). More succinctly, C^1 functions can be approximated in the C^1 sense by trigonometric polynomials. (Hint: Apply Corollary 5.4 to $f'(x)$, and then use the fundamental theorem of calculus, writing $f(x) = f(0) + \int_0^x f'(y)\ dy$).
- Q3 (b). (Optional) Formulate and prove a generalization of the above result to functions f in the class C^k for some $k \geq 0$.
- Q4 (a). Find a sequence of functions f_n for $n = 1, 2, \ldots$ which are each continuous and 2π-periodic, with the property that f_n converges to zero in the mean-square sense, but not in the pointwise sense. (Hint: choose the f_n so that $f_n(x)$ is large at, say, $x = 0$, but has small mean square).
- Q4 (b). Find a sequence of functions f_n for $n = 1, 2, \ldots$ which are each continuous and 2π-periodic, with the property that f_n converges to zero in the pointwise sense, but not in the mean-square sense. (Hint: choose the f_n to be small except on an interval such as $[1/n, 2/n]$, and to have large mean square).

- Q5 (a). Find a sequence of 2π-periodic functions f_n for $n = 1, 2, \ldots$, and another 2π-periodic function f, with the properties that f_n converges to f in mean-square sense, and that each of the f_n is continuous, but that f is discontinuous.

- Q5 (b) (Optional). Repeat (a), but ensure that f_n converges to f in the pointwise *and* the mean-square senses.

- Q6. Let f and g be Riemann-integrable 2π-periodic functions, and let $h = f * g$. Show that the (Dirichlet) partial sums of the Fourier series of h converges absolutely and uniformly to h. (Hint: the following results will be useful: Parseval's identity, Corollary 2.3 of Chapter 2, Proposition 3.1(vi) of Chapter 2, and the Cauchy-Schwarz inequality). Remark: This exercise shows that h is "better" than just being a continuous function (which it is, by Proposition 3.1(v)), since continuous functions do not always enjoy uniformly convergent Dirichlet sums. In fact, h belongs to a regularity class called the *Wiener class*, which is stronger than being continuous, but not as strong as being continuously differentiable. (Being in the Wiener class just means that your Fourier coefficients are absolutely summable).

- Q7. Let f be continuously differentiable and 2π-periodic. Show that

$$\sum_{n=-\infty}^{\infty} (1 + |n|^2)|\hat{f}(n)|^2 < \infty.$$

  (Hint: use Parseval's identity and the identity in the middle of page 43 of the textbook). Conclude that the Fourier series of f converges absolutely and uniformly to f. (Hint: use the Cauchy-Schwarz inequalty and Corollary 2.3 of Chapter 2). Remark: compare this to the last part of Corollary 2.4 of Chapter 2.

- Q8. In this homework question we introduce the notion of *Hölder continuity*, which is a range of intermediate regularity classes between C^0 (continuous functions) and C^1 (continuously differentiable functions).

- Let $0 < \alpha \leq 1$ be a number, and let f be a 2π-periodic function. We say that f is *Hölder continuous of order* α if there exists a number

$M > 0$ such that $|f(x) - f(y)| \leq M|x - y|^\alpha$ for all real numbers x, y. (The class of all Hölder continuous functions of order α is sometimes denoted $C^{0,\alpha}$).

- Q8(a). Show that if f is Hölder continuous of order α, then it is also continuous. (In other words, $C^{0,\alpha}$ is contained in C^0).

- Q8(b). Show that if f is Hölder continuous of order α, and $0 < \beta < \alpha$, then f is also Hölder continuous of order β. (Hint: treat the cases $|x - y| \leq 2\pi$ and $|x - y| > 2\pi$ separately. For the latter case, use the fact that continuous periodic functions are bounded). In other words, $C^{0,\alpha}$ is contained in $C^{0,\beta}$.

- Q8(c). Let f be the function defined by $f(x) := |x|^{1/2}$ if $|x| \leq \pi$, and extended periodicially to the whole real line. Show that f is Hölder continuous of order $1/2$, but is not Hölder continuous of order α for any $1/2 < \alpha < 1$.

- Q8(d). Show that if f is continuously differentiable, then it is also Hölder continuous of order α for every $0 < \alpha \leq 1$. (Hint: prove the $\alpha = 1$ case first, using the Fundamental theorem of Calculus, and then use Q8(b)). In other words, C^1 is contained in $C^{0,\alpha}$. Remark: functions which are Hölder continuous of order 1 (i.e. functions in $C^{0,1}$) are sometimes called *Lipschitz continuous*. Thus all continuously differentiable functions are Lipschitz contintuous.

- Q9. (Optional) Suppose f is Hölder continuous of order α. Show that there exists a constant $M' > 0$ such that $|\hat{f}(n)| \leq M'/|n|^\alpha$ for all $n \neq 0$. (Hint: let $h > 0$ be arbitrary, and consider the functions $\Delta_h f$ defined in the previous homework. Use the Hölder continuity assumption to prove that the Fourier coefficients of $\Delta_h f$ do not exceed $Mh^{\alpha-1}$. Then use Q5(b) of the previous homework and conclude a bound on $\hat{f}(n)$ depending on h. But h is arbitrary; now set $h = 1/|n|$ and see what happens.)

- Q10. Let f and g be Riemann-integrable, 2π-periodic functions. Let $\tilde{g}$ be the function $\tilde{g}(x) := \overline{g(-x)}$.

- Q10(a). What is the relationship between the Fourier coefficients of g and the Fourier coefficients of $\tilde{g}$?

- Q10(b). Show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x)|^2\, dx = (g * \tilde{g})(0).$$

(There are two distinct proofs; one using Parseval's identity, and one using direct computation that avoids all use of the Fourier transform.)

- Q10(c). Show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f * g(x)|^2\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f * \tilde{g}(x)|^2\, dx.$$

(Again, there are two proofs, one via Parseval and one via Q10(b) and Proposition 3.1 of Chapter 2).