Math 132 - Week 6
Textbook sections: 5.1-5.3, 5.5
Topics covered:

- Power Series

- Taylor Series

- Laurent Series

Overview

- In real analysis, many functions $f(x)$ can be expanded as power series:

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots$$

These series expansions are called Maclaurin series if $x_0 = 0$, and Taylor series in general. For instance, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots .$$

Functions which can be written as convergent Taylor series are called *real analytic*.

- These series are important for a number of reasons; for instance they provide quick ways to compute functions such as e^x, providing of course that the series converges.

- In this set of lectures we explore the theory of power series for complex functions. It turns out that every *complex analytic* function $f(z)$ can also be written as a convergent Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

but with complex co-efficients a_n rather than real coefficients. Furthermore, the complex analytic functions are the only functions with convergent Taylor series expansions.

- Now suppose $f(z)$ has a singularity at z_0. Then f cannot be written as a Taylor series. However, it turns out that if the singularity is "isolated", f can still be written as a power series if one also allows *negative* powers of z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

$$+ \frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \ldots$$

These series are known as Laurent series.

- Laurent series allow one to get a better understanding of singularities. Eventually we will be able to classify singularities into several distinct types, such as poles, essential singularities, and removable singularities.

Convergence of sequences - Review

- Let z_n be a sequence $z_1, z_2, \ldots$ of complex numbers. We say that z_n *converges* to a limit z_0 if

$$\lim_{n \to \infty} |z_n - z_0| = 0.$$

  An equivalent definition: z_n converges to z_0 if, for every $\varepsilon > 0$ we can find an $N > 0$ such that $|z_n - z_0| < \varepsilon$ for all $n > N$.

- Example: $1 + i^n/n$ converges to 1 as $n \to \infty$.

- If z_n converges to z_0, we write $\lim_{n \to \infty} z_n = z_0$, or $z_n \to z_0$ as $n \to \infty$. If z_n does not converge to any complex number, we say the sequence is divergent.

- The sequence does not have to start at $n = 1$; one can start a sequence at any finite value of n, and the value of the limit (if it exists) is unchanged.

- Suppose we split z_n into real and imaginary parts $z_n = x_n + iy_n$. Theorem: If z_n converges to $z_0 = x_0 + iy_0$, then x_n converges to x_0 and y_n converges to y_0. Conversely, if x_n converges to x_0 and y_n converges to y_0, then $z_n = x_n + iy_n$ converges to $z_0 = x_0 + iy_0$.
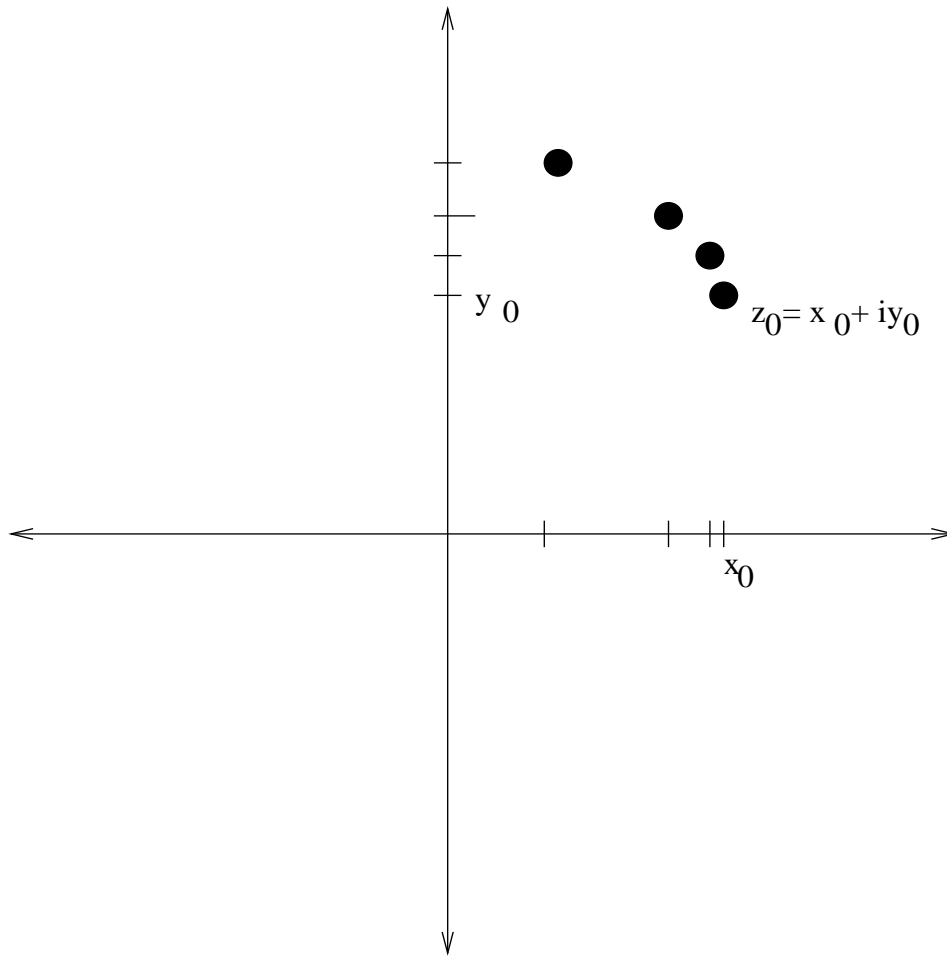
- Thus a sequence converges if and only if its real and imaginary parts converge. In principle, this reduces the study of complex limits to that of real limits, however sometimes it is easier to work with complex limits directly than try to break into real and imaginary parts.

- Some limit laws:

$$\lim_{n\to\infty} z_n \pm w_n = \lim_{n\to\infty} z_n \pm \lim_{n\to\infty} w_n$$

$$\lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} z_n w_n = (\lim_{n\to\infty} z_n)(\lim_{n\to\infty} w_n)$$

$$\lim_{n\to\infty} z_n/w_n = (\lim_{n\to\infty} z_n)/(\lim_{n\to\infty} w_n)$$

  where the last law only works when $\lim_{n\to\infty} w_n \neq 0$.

- The proofs of all these results are boring and will be omitted.

- If z_n converges to z_0, and f is a function which is continuous at z_0, then $f(z_n)$ converges to $f(z_0)$. Thus $\sin(1 + i^n/n)$ converges to $\sin(1)$ as $n \to \infty$, since $\sin(z)$ is always continuous.

- Squeeze test: If $|z_n - z_0| < a_n$ for all n, and a_n is a sequence of positive numbers such that $a_n \to 0$ as $n \to \infty$, then z_n converges to z_0 as $n \to \infty$.

Series

- A series is any expression of the form

$$\sum_{n=0}^{\infty} z_n = z_0 + z_2 + z_3 + \dots$$

  The series can also start at other values than $n = 0$.
  We say that a series converges to a sum S if the partial sums converge to S:

$$\lim_{N \to \infty} \sum_{n=0}^{N} z_n = S.$$

  If the partial sums diverge, we say that the series is divergent. If we don't know whether a series converges or diverges, we call it a formal series.

- If $z_n = x_n + iy_n$ and $\sum_{n=0}^{\infty} z_n = S = X + iY$, then $\sum_{n=0}^{\infty} x_n = X$ and $\sum_{n=0}^{\infty} y_n = Y$ and conversely. In other words, a complex series converges if and only if its real and imaginary parts converge.

- Most series are difficult to compute exactly. However, there is one important series that can be worked out, namely the geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \dots$$

where z is a complex number. The partial sums of this series are

$$\sum_{n=0}^{N} z^n = (1 - z^{N+1})/(1 - z).$$

• When $|z| < 1$, this series converges to $1/(1 - z)$:

$$\sum_{n=0}^{\infty} z^n = 1/(1 - z).$$

For instance, we have

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots = 2.$$

• When $|z| \geq 1$, then the series is divergent. For instance,
$$1 + (2i) + (2i)^2 + (2i)^3 + \ldots$$
diverges (the series does not equal $1/(1 - 2i)$).

• Series laws:

$$\sum_{n=0}^{\infty} (z_n \pm w_n) = \sum_{n=0}^{\infty} z_n \pm \sum_{n=0}^{\infty} w_n.$$

$$\mathrm{Re}(\sum_{n=0}^{\infty} z_n) = \sum_{n=0}^{\infty} \mathrm{Re}(z_n)$$

$$\text{Im}(\sum_{n=0}^{\infty} z_n) = \sum_{n=0}^{\infty} \text{Im}(z_n)$$

$$\sum_{n=0}^{\infty} c z_n = c \sum_{n=0}^{\infty} z_n$$

$$\sum_{n=0}^{\infty} z_n = \sum_{n=0}^{N} z_n + \sum_{n=N+1}^{\infty} z_n$$

- Re-arranging a series can sometimes change the value of a series. As an (admittedly artificial) counterexample, the series

$$1-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{2}-\frac{1}{2}+\frac{1}{4}-\frac{1}{4}+\frac{1}{4}-\frac{1}{4}+\frac{1}{4}-\frac{1}{4}+\frac{1}{4}-\frac{1}{8}\ldots$$

converges to zero; however, if we rearrange the series as

$$1+\frac{1}{2}+\frac{1}{2}-1+\frac{1}{4}+\frac{1}{4}-\frac{1}{2}+\frac{1}{4}+\frac{1}{4}-\frac{1}{2}+\frac{1}{8}+\frac{1}{8}-\frac{1}{4}+\ldots$$

then the series converges to 1. However, this kind of behaviour cannot happen if the series is *absolutely convergent*, which means that $\sum_n |z_n|$ converges.

Convergence tests

- Every absolutely convergent series is convergent, but not vice versa.

- Zero test: if $z_n \not\to 0$ as $n \to \infty$, then $\sum_{n=0}^{\infty} z_n$ is divergent.

- Comparison test: if $|z_n| \le a_n$ for all n and $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty}$ is absolutely convergent.

- Ratio test: suppose $|z_{n+1}|/|z_n|$ converges to a real number L as $n \to \infty$. If $L < 1$, then $\sum_{n=0}^{\infty} z_n$ is absolutely convergent; if $L > 1$, then $\sum_{n=0}^{\infty} z_n$ is divergent. If $L = 1$ or L doesn't exist, then anything can happen.

- There are many other tests, but we will not need them in this course.

Series of functions

- Let $f_0(z), f_1(z), f_2(z), \ldots$ be a sequence of functions. We can form the series

$$\sum_{n=0}^{\infty} f_n(z) = f_0(z) + f_1(z) + f_2(z) + \ldots$$

- This series may converge for some z and diverge for other z. If it does converge for a range of z, then the sum is another function, say $f(z)$:

$$\sum_{n=0}^{\infty} f_n(z) = f(z).$$

- Another way of writing this is

$$\lim_{N \to \infty} \left| f(z) - \sum_{n=0}^{N} f_n(z) \right| = 0.$$

- The series $\sum_{n=0}^{\infty} f_n(z)$ does not always converge at the same rate for each z. For instance, consider the geometric series

$$\sum_{n=0}^{\infty} z^n = 1/(1-z)$$

on the disk $D = \{z : |z| < 1\}$. This series converges for all $z \in D$, but converges much faster when z is

near zero than when z is near the boundary of D. For instance, when $z = 0.1$, then

$$|1/(1-z) - \sum_{n=0}^{100} z^n| \approx 1.1 \times 10^{-100};$$

if $z = 0.9$, then

$$|1/(1-z) - \sum_{n=0}^{100} z^n| \approx 2.6 \times 10^{-4};$$

while if $z = 0.99$, then

$$|1/(1-z) - \sum_{n=0}^{100} z^n| \approx 36.$$

- Because of this, we introduce a stronger notion of convergence, called *uniform convergence.*

- **Definition** If D is a set and $f_n(z)$, $f(z)$ are functions defined on D, we say that $\sum_{n=0}^{\infty} f_n(z)$ *converges uniformly to $f(z)$ on D* if

$$\lim_{N \to \infty} \sup_{z \in D} |f(z) - \sum_{n=0}^{N} f_n(z)| = 0.$$

- For instance, the geometric series $\sum_{n=0}^{\infty}$ does not converge uniformly to $1/(1-z)$ on the disk $\{z : |z| < 1\}$,

but it does converge uniformly to $1/(1-z)$ on the smaller disk $\{z : |z| < 1/2\}$, as we shall see.

- One major advantage of uniform convergence over ordinary convergence is that it allows one to swap sums and integrals:

- **Theorem.** Let D be a domain, and $f_n(z)$ be continuous functions on D. If $\sum_{n=0}^{\infty} f_n(z)$ converges to $f(z)$ uniformly on D, then f is also continuous on D, and for every contour Γ in D we have

$$\int_{\Gamma} f(z)\ dz = \sum_{n=0}^{\infty} \int_{\Gamma} f_n(z)\ dz.$$

- Another way of saying this is that we have

$$\int_{\Gamma} \sum_{n=0}^{\infty} f_n(z)\ dz = \sum_{n=0}^{\infty} \int_{\Gamma} f_n(z)\ dz$$

whenever the sum converges uniformly. For finite sums this claim is obvious, but things are a little subtle for infinite sums. Indeed, without the assumption of uniform convergence the above identity can be false!

- (Proof omitted).

- The definition of uniform convergence is somewhat difficult to work with. Fortunately there is a useful test to determine uniform convergence:

- **Weierstrass M-test.** Let D be a set, and let $f_n(z)$ be functions on D. Suppose that each $f_n(z)$ is bounded by some number a_n, so that $|f_n(z)| \leq a_n$ for all $z \in D$. Suppose also that $\sum_n a_n \leq M$ for some finite number M. Then the series $\sum_{n=0}^{\infty} f_n(z)$ is uniformly convergent on D.

- The proof of this theorem is not difficult, but will be omitted.

- Example: To show that the geometric series converges uniformly on $D = \{z : |z| < 1/2\}$, we write $f_n(z) = z^n$, and observe that $|f_n(z)| \leq 1/2^n$ for all z in D. Since $\sum_n 1/2^n$ converges, it is bounded by some number M, and we are done.
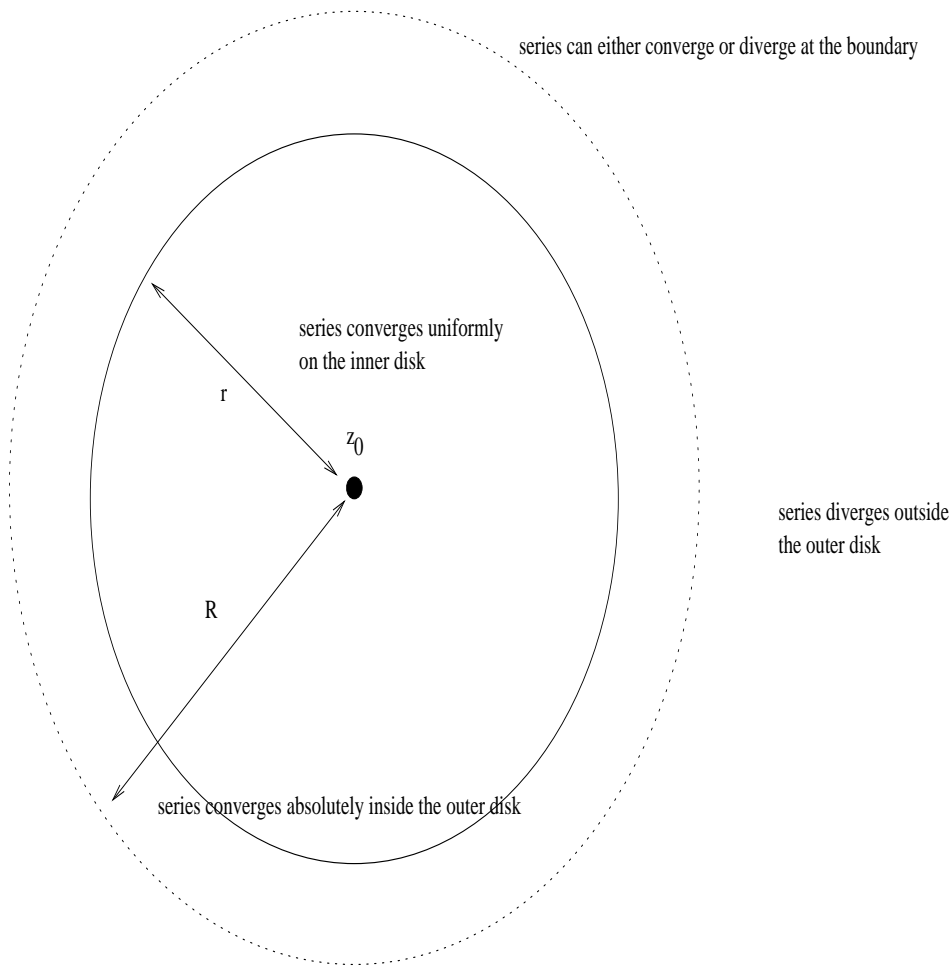
Power series

- A power series is any expression of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

where the a_n are complex co-efficients, and z_0 is a complex number.

- The series obviously converges when $z = z_0$, and can converge for other values of z too, but does not necessarily converge for all z. However, the set of z for which the series converges is always a disk of some sort:

- **Theorem.** Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series. Then there exists a number $0 \leq R \leq \infty$ such that the series converges absolutely for $|z - z_0| < R$ and diverges for $|z - z_0| > R$. (For $|z - z_0| = R$ anything can happen). Also, for any $0 < r < R$, the series converges uniformly on $\{z : |z - z_0| \leq r\}$.

series can either converge or diverge at the boundary

series converges uniformly
on the inner disk

r

z_0

series diverges outside
the outer disk

R

series converges absolutely inside the outer disk

- The number R is called the *radius of convergence* of the series. The set where z converges is called is called the *disk of convergence*. This disk can be a single point (when $R = 0$) or the entire plane (when $R = \infty$) or something in between.

- (Proof omitted).

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- In practice, the radius of convergence is computed using the ratio test. For instance, consider the series

$$\sum_{n=0}^{\infty} \frac{nz^n}{2^n}.$$

We compute the limiting ratio

$$L = \lim_{n \to \infty} \frac{\left|(n+1)z^{n+1}/2^{n+1}\right|}{\left|nz^n/2^n\right|}$$

$$= \lim_{n \to \infty} \frac{n+1}{n} \frac{|z|}{2} = \frac{|z|}{2}.$$

Thus we have convergence when $|z| < 2$ and divergence when $|z| > 2$, so the radius of convergence is 2. (The behaviour at the boundary $|z| = 2$ is more delicate. In this particular case, one has divergence at $|z| = 2$ thanks to the zero test).

- A series may have a zero radius of convergence. For instance, the series

$$\sum_{n=0}^{\infty} n!z^n = 1 + z + 2z^2 + 6z^3 + \ldots$$

turns out to only converge when $z = 0$.

Taylor's formula

- We now show that any complex analytic function can be written as a power series.

- **Taylor's formula** Let f be a function which is analytic on a disk $\{z : |z - z_0| \leq R\}$. Then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = f(z_0)+f'(z_0)(z-z_0)+f''(z_0)\frac{(z - z_0)^2}{2!}+\ldots$$

converges to $f(z)$ on the disk $\{z : |z - z_0| < R\}$.

- This series is called the *Taylor series* of f at z_0. If $z_0 = 0$, this series is called the *Maclaurin series* of f.

- **Proof** We first do a translation trick which effectively moves z_0 to the origin. Let $g(z)$ denote the function $g(z) = f(z + z_0)$. Then $g(0) = f(z_0)$, $g'(0) = f'(z_0)$, etc, and g is analytic on the disk $\{z : |z| \leq R\}$. We now need to show that the power series

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}z^n = g(0) + g'(0)z + g''(0)\frac{z^2}{2!} + \ldots$$

converges to $g(z)$ on the disk $\{z : |z| < R\}$.

- The proof of this is very different from the real analytic case, and goes through the Cauchy integral formula. Let γ be the circle $\{|z| = R\}$ traversed once anti-clockwise. We have

$$g(z) = \frac{1}{2\pi i} \oint_\gamma \frac{g(w)}{w - z} \, dw$$

for all $|z| < R$.

- We want to write this as a power series in z. The trick is to use the geometric series formula. We rewrite

$$\frac{1}{2\pi i} \oint_\gamma \frac{g(w)}{w - z} \, dw = \frac{1}{2\pi i} \oint_\gamma \frac{g(w)}{w} \frac{1}{1 - z/w} \, dw.$$

Since w is in γ, $|w| = R$, and so $|z/w| < 1$. We can then use the geometric series formula to obtain

$$\frac{1}{2\pi i} \oint_\gamma \frac{g(w)}{w - z} \, dw = \frac{1}{2\pi i} \oint_\gamma \frac{g(w)}{w} (1 + \frac{z}{w} + \frac{z^2}{w^2} + \dots) \, dw$$

$$= \oint_\gamma \sum_{n=0}^{\infty} \frac{g(w) z^n}{w^{n+1}} \, dw.$$

If the series $\frac{g(w) z^n}{w^{n+1}}$ converges uniformly (this can be done using the M-test, but we'll omit this), we can interchange the sum and integral to obtain

$$g(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_\gamma \frac{g(w) z^n}{w^{n+1}} \, dw.$$

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However, from the generalized Cauchy integral formula we have

$$\oint_\gamma \frac{g(w)}{w^{n+1}} \, dw = \frac{2\pi i}{n!} g^{(n)}(0).$$

Putting the two facts together we see that

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n$$

as desired.

$\square$

Some standard Taylor-Maclaurin series:

- The exponential, sine and cosine Maclaurin series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \dots$$

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$$

all have an infinite radius of convergence. Note that from these three expansions one can prove the complex form of Euler's formula:

$$e^{iz} = \cos(z) + i\sin(z).$$

- The Maclaurin series for $1/(1-z)$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots$$

converges for $|z| < 1$ but diverges for $|z| \geq 1$.

- The Maclaurin series for $\mathrm{Log}(1-z)$

$$\mathrm{Log}(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} = -z - \frac{z^2}{2} - \frac{z^3}{3} - \dots$$

converges for all $|z| \leq 1$ except $z = 1$, but diverges for all other z.

- Using the standard series one can work out what happens for modified series fairly easily. For instance, by substituting $2z^2$ for z in the expansion of $1/(1-z)$ we see that

$$\frac{1}{1 - 2z^2} = \sum_{n=0}^{\infty} 2^n z^{2n}$$

  converges for $|2z^2| < 1$, i.e. for $|z| < 1/\sqrt{2}$. One can also use Taylor's formula to do this, of course, but it's more work that way.

- Another example: if we want to find the Taylor series of e^z around $z_0 = 1$ (rather than $z_0 = 0$), we can write

$$e^z = e^{z-1+1} = e e^{z-1}$$

  and then use the previous formula to obtain

$$e^z = \sum_{n=0}^{\infty} e \frac{(z-1)^n}{n!}$$

  for all z.

- Now consider the problem of finding the Maclaurin expansion of $\frac{1}{(1-z)^2}$. We can do this by squaring the

series for $1/(1-z)$:

$$\frac{1}{(1-z)^2} = (1 + z + z^2 + \ldots)(1 + z + z^2 + \ldots).$$

When $|z| < 1$, both series are absolutely convergent, and we can rearrange them however we want and still get the same answer. If we multiply out all the terms and then simplify, we obtain

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \ldots$$

when $|z| < 1$.

- More generally, if we know the Taylor series of $f(z)$ around z_0, and the Taylor series of $g(z)$ around z_0, then we can multiply the two together to obtain the series for $f(z)g(z)$ around z_0.

- Addition and subtraction of Taylor series is also straight-forward. Division is a little trickier. For instance, suppose we want to find the Maclaurin series of $\sec(z) = 1/\cos(z)$. Since $\sec(z)$ is analytic in the disk $|z| < \pi/2$, we know that there is *some* Maclaurin series for $\sec(z)$ which has a radius of convergence of at least $\pi/2$:

$$\sec(z) = a_0 + a_1 z + a_2 z^2 + \ldots$$

Since $\cos(z)\sec(z) = 1$, we thus have

$$(1 - \frac{z^2}{2!} + \ldots)(a_0 + a_1 z + a_2 z^2 + \ldots) = 1.$$

Multiplying out the left-hand side when $|z| < \pi/2$, and comparing co-efficients, we obtain

$$a_0 = 1$$
$$a_1 = 0$$
$$a_2 - \frac{a_0}{2} = 0$$
$$a_3 - \frac{a_1}{2} = 0$$

etc. which allows us to solve for the unknown co-efficients a_i:

$$\sec(z) = 1 + \frac{1}{2}z^2 - \frac{1}{12}z^4 + \ldots$$

- Another way to do this is by long division, dividing $1 - \frac{z^2}{2!} + \ldots$ into 1. One follows the essentially same procedure one uses to long divide one polynomial into another, except that one starts from the lowest degree polynomial and works one's way upwards.

- Division works well as long as the denominator has a non-zero constant co-efficient (i.e. the denominator doesn't vanish when $z = z_0$). When it does vanish one has to be slightly more careful, as we shall see.

Integrating a power series

● Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

be a power series around z_0 with radius of convergence R, and suppose that γ is a contour from z_1 to z_2 in the disk $\{z : |z - z_0| \le r\}$ for some $0 < r < R$.
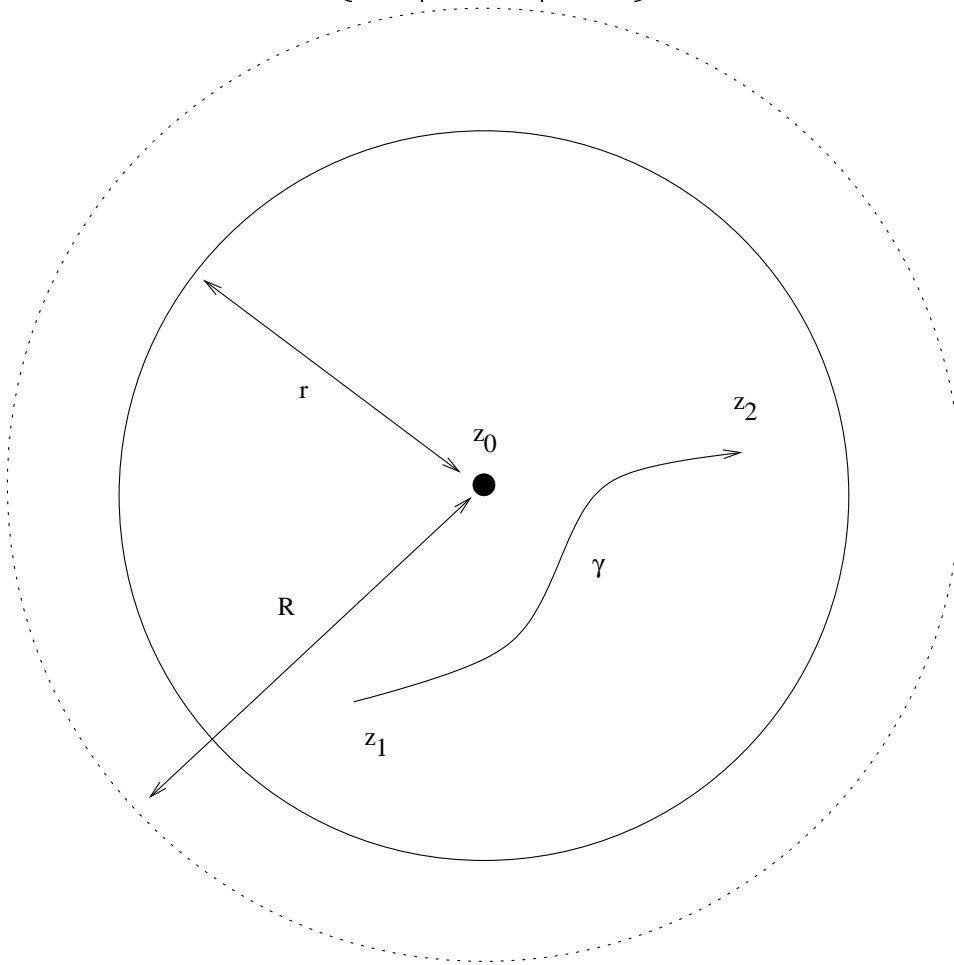
We want to compute
$$\int_\gamma f(z)\ dz.$$
We expand this as
$$\int_\gamma \sum_{n=0}^{\infty} a_n (z - z_0)^n\ dz.$$
Since the power series converges uniformly on γ, we can interchange the sum and integral
$$\sum_{n=0}^{\infty} \int_\gamma a_n (z - z_0)^n\ dz.$$
By the fundamental theorem of calculus, this becomes
$$\sum_{n=0}^{\infty} a_n \frac{(z - z_0)^{n+1}}{n + 1}\Big|_{z_1}^{z_2}.$$
In other words, we see that
$$\sum_{n=0}^{\infty} a_n \frac{(z - z_0)^{n+1}}{n + 1}$$
is an anti-derivative of $\sum_{n=0}^{\infty}$.

- As an example, we know that the series
$$-1/(1 - z) = -1 - z - z^2 - z^3 - \dots$$

which converges for $|z| < 1$, has an anti-derivative

$$-z - z^2/2 - z^3/3 - \ldots$$

which also converges for $|z| < 1$.

- The function $\mathrm{Log}(1 - z)$ is also an anti-derivative of $-1/(1 - z)$ on the disk $|z| < 1$. ($\mathrm{Log}(1 - z)$ is not analytic when $1-z$ is zero or a negative real, but that does not happen inside this disk). Thus $\mathrm{Log}(1 - z)$ can only differ by a constant from the above power series on this disk. When $z = 0$ both functions are zero, so in fact the two functions are equal:

$$\mathrm{Log}(1 - z) = -z - z^2/2 - z^3/3 - \ldots$$

for $|z| < 1$.

- (If two functions have the same derivative on the same domain, then they must differ by a constant on that domain.)

Radius of convergence of a power series

- We have just shown that every power series has an anti-derivative inside its disk of convergence. From previous notes we know that if a function has an anti-derivative on a domain, it must be analytic on this domain (because the integral around every closed contour is zero, and one can apply Morera's theorem). Thus every power series is analytic on its disk of convergence.

- On the other hand, Taylor's formula says that every analytic function on a disk has a power series that converges on that disk. If one combines the two results, one can work out exactly what the radius of convergence of a Taylor series is.
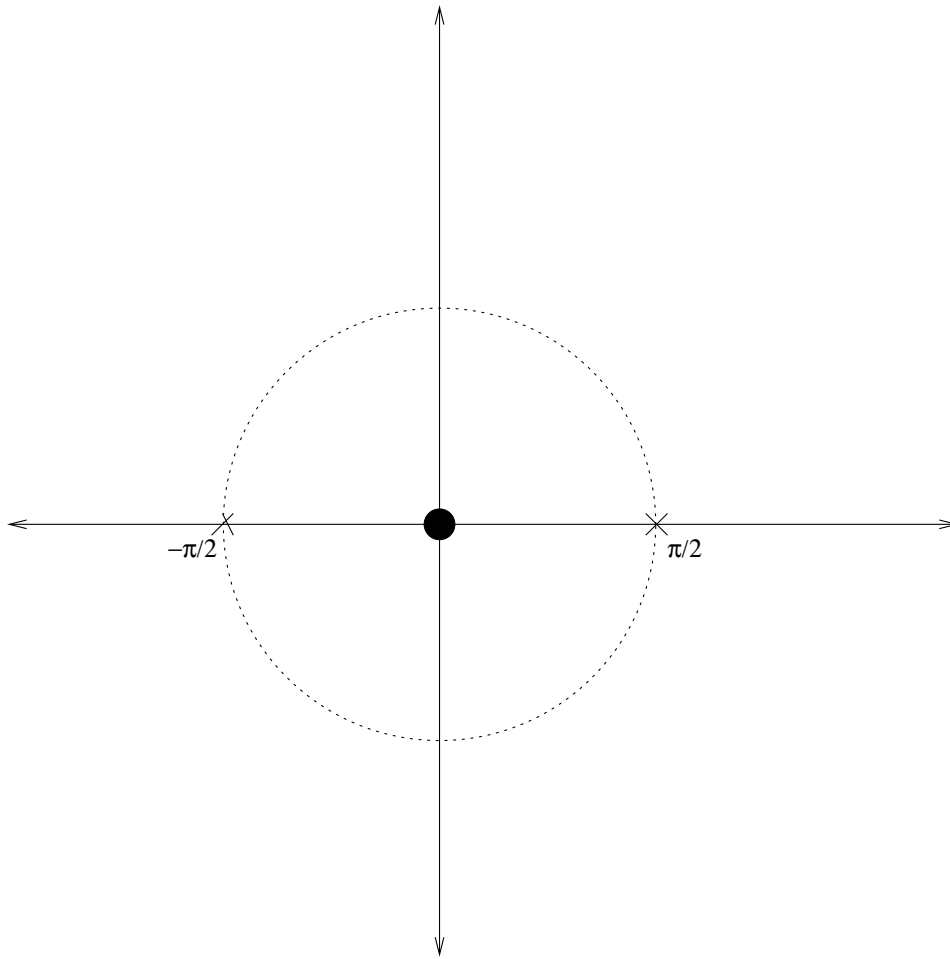
- As an example, consider the Maclaurin expansion of $f(z) = e^z/\cos(z)$. Without actually evaluating any coefficients, we know that the radius of convergence of this series is at least $\pi/2$, because f is analytic on the disk of radius $\pi/2$ around the origin. And we know that the radius of convergence cannot exceed $\pi/2$, since that would imply that the power series is

analytic on a disk of radius greater than $\pi/2$, which is a contradiction since f has goes to infinity at $\pm\pi/2$.

- As a general rule of thumb, the radius of convergence of a Taylor expansion of f at z_0 is equal to the distance from z_0 to the nearest singularity of f. This is not always the case however, because some singularities are "removable". As a rather trivial example, consider the function

$$f(z) = z/z.$$

Strictly speaking, this function has a singularity at 0, and outside of zero this function equals 1. If one performed a Taylor expansion at $z_0 = 1$, one would obtain

$$f(z) = 1 + 0(z - 1) + 0(z - 1)^2 + \dots$$

which of course has an infinite radius of convergence, despite f having a singularity 1 unit away from z_0. We'll discuss the difference between removable and non-removable singularities later in the course.

Differentiating power series

- We have just shown that power series can be integrated inside their disk of convergence. Now we show that we can also differentiate these series in their disk of convergence.

- **Theorem.** Suppose that
$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$
has a radius of convergence greater than R. Then one has
$$f'(z) = \sum_{n=1}^{\infty} a_n n(z-z_0)^{n-1} = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \ldots$$
whenever $|z - z_0| < R$.

- **Proof.** Let γ be the circle of radius R around z_0 traversed once anti-clockwise, and let z be a point such that $|z - z_0| < R$, so that z is inside Γ. By the generalized Cauchy integral formula, we have
$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)\,dw}{(w-z)^2}.$$
In other words, we have
$$f'(z) = \frac{1}{2\pi i} \int_\gamma \sum_{n=0}^{\infty} \frac{a_n(z-z_0)^n\,dw}{(w-z)^2}.$$

The power series is uniformly convergent, so we may interchange the sum and integral. (The additional $1/(w-z)^2$ factor is bounded on γ and does not cause a difficulty). We thus have

$$f'(z) = \sum_{n=0}^{\infty} a_n \frac{1}{2\pi i} \int_\gamma \frac{(z-z_0)^n \, dw}{(w-z)^2}.$$

By the generalized Cauchy integral formula applied to $f(w) = (w-z_0)^n$, we have

$$n(z-z_0)^{n-1} = \frac{1}{2\pi i} \int_\gamma \frac{(z-z_0)^n \, dw}{(w-z)^2}.$$

Combining these two equations we get the theorem.

• As an example, suppose we want to find the Maclaurin series of $1/(1-z)^2$. One way to do this is to square $1/(1-z)$ as we did earlier. Another way is to begin with the series for $1/(1-z)$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots$$

for $|z| < 1$, and then differentiate both sides using the above theorem to get

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \dots$$

Uniqueness of Taylor series

- We can now clear up one small issue: is it possible for a single function to have two distinct power series which both converge around z_0? In other words, can we have

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

for all $|z - z_0| < R$, with the a_n co-efficients being distinct from the b_n co-efficients?

- The answer is no. To see this, first set $z = z_0$ in the above equation to get $a_0 = b_0$. Then differentiate the series once to get

$$a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + \ldots = b_1 + 2b_2(z - z_0) + 3b_3(z - z_0)^2 + \ldots$$

setting $z = z_0$ again we get $a_1 = b_1$. If we keep iterating this procedure we see that all the co-efficients have to match.

- As a corollary, we see that if we know that two power series agree, then we can equate their coefficients.

Laurent series

- We have seen that if f is an analytic function at z_0, then f can be written as a Taylor expansion around z_0, and the radius of convergence should (in most cases) be equal to the distance from f to the nearest singularity. When z_0 is far away from any singularities these Taylor series are quite acceptable. But if one moves z_0 too close to a singularity, or even onto a singularity, then things get worse. For instance, the function $1/(1-z)$ has a Taylor series at $z_0 = 0$

$$1/(1-z) = 1 + z + z^2 + \ldots$$

with radius of convergence 1. At $z_0 = 1/2$, the series becomes

$$1/(1-z) = 2 + 4(z - 1/2) + 8(z - 1/2)^2 + \ldots$$

and has a radius of convergence of only $1/2$. At $z_0 = 9/10$, the series becomes

$$1/(1-z) = 10 + 100(z - 9/10) + 1000(z - 9/10)^2 + \ldots$$

and the radius of convergence has shrunk to $1/10$. At $z_0 = 1$, there is no Taylor series at all.

- Fortunately, there is another type of series expansion that converges on a reasonably large set even when

there are singularities at or near to z_0. These are known as *Laurent* series.

- Formally, a Laurent series is just a double power series, using both negative and positive powers of $(z - z_0)$. In other words, anything of the form

$$\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

$$+ \frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \dots$$

- In particular, every Taylor series is a Laurent series, but not conversely.

- As we've seen, Taylor series always converge inside a disk. Laurent series can do this, but more often they converge in an annulus. For instance, consider the Laurent series

$$\sum_{n=-\infty}^{\infty} \frac{z^n}{2^{|n|}} = 1 + \frac{z}{2} + \frac{z^2}{4} + \dots + \frac{1}{2z} + \frac{1}{4z^2} + \dots$$

The positive power half of the series,

$$1 + \frac{z}{2} + \frac{z^2}{4} + \dots$$

is a plain old Taylor series, and converges in the disk $|z| < 2$. The negative power half of the series

$$\frac{1}{2z} + \frac{1}{4z^2} + \ldots$$

can be rewritten as a Taylor series if we make the substitution $w = 1/z$:

$$\frac{w}{2} + \frac{w^2}{4} + \ldots$$

one can show that this series converges when $|w| < 2$, i.e. when $|z| > 1/2$. Putting the two together, we see that the full Laurent series converges in the annulus $1/2 < |z| < 2$.

- Some Laurent series do not converge anywhere at all. For instance, consider the series

$$\sum_{n=-\infty}^{\infty} 2^{|n|} z^n = 1 + 2z + 4z^2 + \ldots + \frac{2}{z} + \frac{4}{z^2} + \ldots .$$

The positive power half of the series converges when $|z| < 1/2$, while the negative power half converges when $|z| > 2$. There is no complex number z such that both parts of the series simultaneously converge.

- Generally, we have

- **Proposition.** Let $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ be a Laurent series. Then there are numbers $0 \leq r, R \leq \infty$ such that the series converges absolutely to an analytic function for $r < |z - z_0| < R$ and diverges for $|z - z_0| < r$ or $|z - z_0| > R$. (It is possible for R to be less than r, in which case the series never converges). If $r < r' < R' < R$, then the series converges uniformly on the smaller annulus $\{z : r' \leq |z - z_0| \leq R'\}$.

- R and r are sometimes called the outer and inner radii of convergence.

- This is proved by breaking up the Laurent series into a power series in $(z - z_0)$ and a power series in $1/(z - z_0)$.

- Just as every function which is analytic on a disk can be written as a power series, every function which is analytic on an annulus can be written as a Laurent series.

- Example: Suppose we want to find the Laurent series of $1/(1 - z)$ around 0 in the annulus $1 < |z| < \infty$. The standard Taylor series

$$1/(1 - z) = 1 + z + z^2 + \dots$$

does not converge here, and we need to find another series. The trick is to write things in terms of $1/z$ instead of z. If we write

$$\frac{1}{1-z} = \frac{1}{z(1/z-1)} = \frac{-1}{z}\frac{1}{1-1/z},$$

we note that the geometric series formula works for $1/z$ since $|1/z| < 1$:

$$\frac{1}{1-1/z} = 1 + 1/z + 1/z^2 + \ldots.$$

Putting the two together we obtain the Laurent expansion of $1/(1-z)$ around 0 in the annulus $1 < |z| < \infty$:

$$\frac{1}{1-z} = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots$$

- A more complicated example: suppose we want to work out all the Laurent expansions of

$$\frac{1}{(1-z)(2-z)}$$

around 0. We can break this up using partial fractions as

$$\frac{1}{(1-z)(2-z)} = \frac{1}{1-z} - \frac{1}{2-z}.$$

As observed before, the function $1/(1 - z)$ has an expansion of

$$\frac{1}{1 - z} = 1 + z + z^2 + \ldots$$

when $|z| < 1$, and

$$\frac{1}{1 - z} = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \ldots$$

when $|z| > 1$. Similarly (exercise!) we have

$$\frac{1}{2 - z} = \frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \ldots$$

when $|z| < 2$, and

$$\frac{1}{2 - z} = -\frac{1}{z} - \frac{2}{z^2} - \frac{4}{z^3} - \ldots$$

when $|z| < 2$. Putting all this together we get three Laurent series. When $|z| < 1$, we get

$$\frac{1}{(1 - z)(2 - z)} = \frac{1}{2} + \frac{3z}{4} + \frac{7z^2}{8} + \ldots$$

When $1 < |z| < 2$, we get

$$\frac{1}{(1 - z)(2 - z)} = -\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \ldots$$
$$-\frac{1}{z} - \frac{1}{z^2} - \ldots$$

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When $|z| > 2$, we get

$$\frac{1}{(1-z)(2-z)} = \frac{1}{z^2} + \frac{3}{z^3} + \frac{7}{z^4} + \dots$$

- Another example: Consider the problem of evaluating $e^{1/z}$ around 0. This function has a singularity at 0 (quite a bad one, actually!) and so has no chance of having a Taylor series around 0. However, it is still easy to find a Laurent series in the region $0 < |z| < \infty$ (i.e. all points of the complex plane except the origin). Simply start with the formula

$$e^z = 1 + z + z^2/2! + \dots$$

valid for all z, and replace z by $1/z$ to obtain the desired Laurent expansion

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \dots$$

- More generally, we have

- **Theorem** Suppose that f is analytic on an annulus $r \le |z - z_0| \le R$ for some $R > r$. Then there exists a unique Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

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for f which converges on the annulus $r < |z - z_0| < R$.

- **Proof.** By the same translation trick used to prove the Taylor series formula, we can make $z_0 = 0$.

- Let Γ_R denote the circle $|z| = R$ traversed once anti-clockwise, and let Γ_r denote the circle $|z| = r$ traversed once anti-clockwise. Let z be a point in the annulus $r < |z| < R$. If we knew that f was analytic inside the hole of the annulus, we could use Cauchy's integral formula to write

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{f(w)}{w - z} \, dw,$$

however we cannot assume this. Instead, we can show that
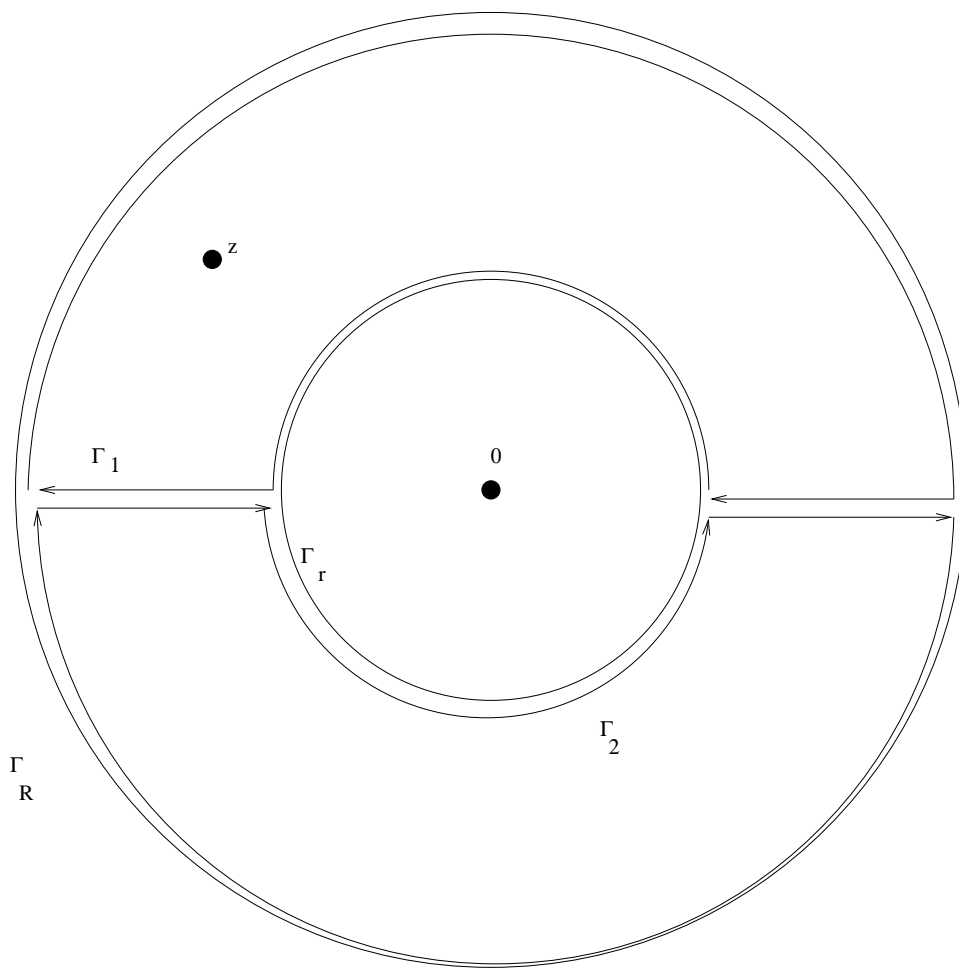
$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{f(w)}{w - z} \, dw - \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(w)}{w - z} \, dw.$$

To see this, we make two closed contours Γ_1 and Γ_2 such that Γ_1 goes anti-clockwise around $f(z)$ and Γ_2

doesn't. From the CIF and Cauchy-Goursat we have

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)}{w - z} \, dw + \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(w)}{w - z} \, dw,$$

and the claim follows by rearranging the contours.

- We need to write the RHS as a Laurent series around 0. Let's first look at the first integral. If w is in Γ_1, then $|w| = R$, so $|z/w| < 1$. Hence we can use the geometric series formula for z/w:

$$\frac{1}{w - z} = \frac{1}{w} \frac{1}{1 - z/w} = \frac{1}{w} + \frac{z}{w^2} + \frac{z^2}{w^3} + \dots.$$

We therefore have

$$\int_{\Gamma_1} \frac{f(w)}{w - z} \, dw = \int_{\Gamma_1} \sum_{n=0}^{\infty} \frac{f(w)}{w^{n+1}} z^n \, dw.$$

The series happens to converge uniformly (proof omitted), so we get

$$\int_{\Gamma_1} \frac{f(w)}{w - z} \, dw = \sum_{n=0}^{\infty} \left(\int_{\Gamma_1} \frac{f(w)}{w^{n+1}} \, dw \right) z^n.$$

This is a power series in positive powers of z.

- Now let's look at the Γ_2 integral. Now $|w| = r$, so that $|w/z| < 1$. We can now use the geometric series

formula for w/z:

$$\frac{1}{w-z} = -\frac{1}{z}\frac{1}{1-w/z} = -\frac{1}{z} - \frac{w}{z^2} - \frac{w^2}{z^3} - \cdots$$

We therefore have

$$\int_{\Gamma_2} \frac{f(w)}{w-z} \, dw = \int_{\Gamma_2} -\sum_{n=0}^{\infty} \frac{f(w)}{z^{n+1}} w^n \, dw.$$

We can prove uniform convergence as before (exercise!), and so we get

$$\int_{\Gamma_2} \frac{f(w)}{w-z} \, dw = \sum_{n=0}^{\infty} \frac{-\int_{\Gamma_2} f(w) w^n \, dw}{z^{n+1}}.$$

This is a power series in negative powers of z. Combining the two formulae together see that we have written f as a Laurent series on this annulus.

- Now we show that the Laurent series is unique. Suppose we have two Laurent series which both converge uniformly on some annulus $r \leq |z| \leq R$:

$$\sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n=-\infty}^{\infty} b_n z^n.$$

Integrating both series around Γ_R, and using uniform convergence to interchange the sum and integral, we

obtain

$$\sum_{n=-\infty}^{\infty} a_n \int_{\Gamma_R} z^n \, dz = \sum_{n=-\infty}^{\infty} b_n \int_{\Gamma_R} z^n \, dz.$$

Except for z^{-1}, the functions z^n all have anti-derivatives on Γ_R (namely $z^{n+1}/(n+1)$) and so their integral on a closed contour is just 0. So we're left with

$$a_{-1} \int_{\Gamma_R} \frac{dz}{z} = b_{-1} \int_{\Gamma_R} \frac{dz}{z}.$$

- By the Cauchy Integral formula, or by playing around with complex logarithms, one has $\int_{\Gamma_R} \frac{dz}{z} = 2\pi i \neq 0$. So we can cancel this integral and obtain $a_{-1} = b_{-1}$. To show that the other co-efficients match, we multiply or divide both series by some power of z and repeat the above argument. For instance, if we multiply both sides by z, then it is the a_{-2} and b_{-2} co-efficients which don't cancel when one integrates on Γ_R.

  □

- This theorem shows that Laurent series exist, but do not give an easy recipe for finding the coefficients of the series. In practice one has to use ad hoc techniques for finding Laurent series, like the ones men-

tioned above. Some Laurent series are quite difficult; for instance, working out the Laurent expansion of $e^{1/z}\cos(z)$ for $0 < |z| < \infty$ is very difficult to work out, as when multiplying the Laurent series for $e^{1/z}$ (which contains infinitely many powers of $1/z$) against the Taylor series for $\cos(z)$ (which contains infinitely many powers of z) one gets an infinite number of terms to collect for every power of z.